Homogeneous spaces
B. Komrakov seminar

FOUR–DIMENSIONAL PSEUDO–RIEMANNIAN HOMOGENEOUS SPACES. CLASSIFICATION OF REAL PAIRS

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CONTENTS

Introduction

We consider classification of lower-dimensional homogeneous spaces an immediate continuation and global version of classification results obtained by Sophus Lie. Two-dimensional homogeneous spaces were classified locally by Sophus Lie [L1] and globally by G.D. Mostow [M]. (See also preprint [KTD], where the complete classification of two-dimensional homogeneous spaces, both locally and globally, is presented.) S. Lie also obtained some results in classification of three-dimensional homogeneous spaces and described all subalgebras in the Lie algebra $\mathfrak{so}(4,\mathbb{C})$ (in terms of vector fields). A detailed account of these classifications can be found in [L2]. The local classification of all three-dimensional isotropically-faithful homogeneous spaces was obtained in [KT], and the classification (local and global) of all two- and three-dimensional pseudo-Riemannian isotropically-faithful homogeneous spaces was given in [DK].

The problem of classification of four-dimensional pseudo-Riemannian homogeneous spaces is interesting from the point of view of both geometry and physics, and not only in the case of signature $(1,3)$ (spaces of relativity theory) but also in the case of signature $(2,2)$ (twistors).

Let $(\overline{H},M)$ be a homogeneous space, $H=\overline{H}_x$ the stabilizer of an arbitrary point $x\in M$, and $(\bar{\mathfrak{h}},\mathfrak{h})$ the pair of Lie algebras corresponding to the pair $(\overline{H},H)$ of Lie groups.

Lemma. *Suppose that the homogeneous space $(\overline{H},M)$ admits an invariant pseudo-Riemannian metric. Then the isotropic representation of the pair $(\bar{\mathfrak{h}},\mathfrak{h})$*

$$\rho\colon \mathfrak{h}\to \mathfrak{gl}(\bar{\mathfrak{h}}/\mathfrak{h}),\qquad \rho(x)(\bar{x}+\mathfrak{h})=[x,\bar{x}]+\mathfrak{h}\quad (x\in\mathfrak{h},\ \bar{x}\in\bar{\mathfrak{h}})$$

is faithful. Moreover, there exists a basis of $\bar{\mathfrak{h}}/\mathfrak{h}$ such that $\rho(\mathfrak{h})$ lies in one of the following Lie algebras: $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$, or $\mathfrak{so}(2,2)$, which are the real forms of the complex Lie algebra $\mathfrak{so}(4,\mathbb{C})$.

In accordance with this, we divide solution of our problem into the following parts:

(1) We find (up to conjugation) all possible forms the subalgebra $(\rho(\mathfrak{h}))^{\mathbb{C}}=\rho^{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}})$ can assume. This is equivalent to classifying (up to conjugation) subalgebras $\mathfrak{p}$ in the Lie algebra $\mathfrak{so}(4,\mathbb{C})$.

(2) For each subalgebra $\mathfrak{p}$ obtained in (1), we find (up to equivalence of pairs) all complex pairs $(\bar{\mathfrak{g}},\mathfrak{g})$ such that the subalgebra $\rho^{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}})$ is conjugate to $\mathfrak{p}$ (here $\mathfrak{g}=\mathfrak{h}^{\mathbb{C}}$).

(3) For each complex pair $(\bar{\mathfrak{g}},\mathfrak{g})$, we find (up to equivalence of pairs) all its real forms $(\bar{\mathfrak{g}}^{\sigma},\mathfrak{g}^{\sigma})$, where σ is an anti-involution of the pair $(\bar{\mathfrak{g}},\mathfrak{g})$.

(4) For each real pair obtained in (3), we construct all (up to isomorphism) corresponding homogeneous spaces.

The results of the first part of our work (the classification of the complex pairs) was obtained in [K1] (for a summary of results see [K]). We recall this classification in Chapter I.

This paper presents the results of the second part (the classification of the real pairs) of our work devoted to classification of four-dimensional homogeneous spaces with an invariant pseudo-Riemannian metric of arbitrary signature. We give this classification in Chapter II. A similar classification for the case of Riemannian metric can be found in [I].

CHAPTER I

COMPLEX PSEUDO–RIEMANNIAN PAIRS

1. Classification of subalgebras in the Lie algebra $\mathfrak{so}(4,\mathbb{C})$

Preliminaries:

1. For the sake of simplicity instead of the standard notation for a subalgebra of $\mathfrak{so}(4,\mathbb{C})$ such as

$$\mathfrak{g}=\left\{\left.\begin{pmatrix} x & 0 & 0 & 0\\ 0 & \lambda x & 0 & 0\\ 0 & 0 & -x & 0\\ 0 & 0 & 0 & -\lambda x\end{pmatrix}\right| x\in\mathbb{C}\right\},$$

where $|\lambda|<1,\ -\frac{\pi}{2}<\arg\lambda\leqslant\frac{\pi}{2}$ or $|\lambda|=1,\ 0\leqslant\arg\lambda\leqslant\frac{\pi}{2}$ we use the following notation:

$$\mathfrak{g}=\begin{pmatrix} x & 0 & 0 & 0\\ 0 & \lambda x & 0 & 0\\ 0 & 0 & -x & 0\\ 0 & 0 & 0 & -\lambda x\end{pmatrix}$$

$$|\lambda|<1,\ -\tfrac{\pi}{2}<\arg\lambda\leqslant\tfrac{\pi}{2}\text{ or }|\lambda|=1,\ 0\leqslant\arg\lambda\leqslant\tfrac{\pi}{2}.$$

Here we imply that variables denoted by Latin letters run through $\mathbb{C}$ and that parameters are denoted by small Greek letters.

2. To refer to subalgebras determined in Theorem 1 we use the following notation:

$$d.n,$$

where d is the dimension of the subalgebra; n is the number of the subalgebra in Theorem 1.

Theorem 1. *Any non-zero subalgebra of the Lie algebra $\mathfrak{so}(4,\mathbb{C})$ is conjugate (with respect to $GL(4,\mathbb{C})$) to one and only one of the following subalgebras:*

$$\underline{\dim\mathfrak{g}=1}$$

1.1 $\begin{pmatrix} x & 0 & 0 & 0\\ 0 & \lambda x & 0 & 0\\ 0 & 0 & -x & 0\\ 0 & 0 & 0 & -\lambda x\end{pmatrix}$ 1.2 $\begin{pmatrix} x & x & 0 & 0\\ 0 & x & 0 & 0\\ 0 & 0 & -x & 0\\ 0 & 0 & -x & -x\end{pmatrix}$

$|\lambda|<1,\ -\frac{\pi}{2}<\arg\lambda\leqslant\frac{\pi}{2}$ or $|\lambda|=1,\ 0\leqslant\arg\lambda\leqslant\frac{\pi}{2}$

1.3 $\begin{pmatrix} 0 & 0 & x & 0\\ 0 & 0 & 0 & x\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$ 1.4 $\begin{pmatrix} 0 & x & 0 & 0\\ 0 & 0 & x & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$

$\underline{dim\ \mathfrak{g} = 2}$

2.1 $\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -y \end{pmatrix}$ 2.2 $\begin{pmatrix} x & y & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & -\lambda x \end{pmatrix}$

$|\lambda| < 1$ or $|\lambda| = 1,\ 0 \leqslant \arg\lambda \leqslant \pi$

2.3 $\begin{pmatrix} x & y & 0 & x \\ 0 & -x & -x & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & x \end{pmatrix}$ 2.4 $\begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

2.5 $\begin{pmatrix} 0 & x & 0 & y \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \end{pmatrix}$

$\underline{dim\ \mathfrak{g} = 3}$

3.1 $\begin{pmatrix} x & z & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -z & -y \end{pmatrix}$ 3.2 $\begin{pmatrix} x & y & 0 & z \\ 0 & \lambda x & -z & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & -\lambda x \end{pmatrix}$

$\operatorname{Re}\lambda > 0$, or $\operatorname{Re}\lambda = 0,\ \operatorname{Im}\lambda \geqslant 0$

3.3 $\begin{pmatrix} 0 & y & 0 & z \\ 0 & x & -z & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -y & -x \end{pmatrix}$ 3.4 $\begin{pmatrix} x & y & 0 & 0 \\ z & -x & 0 & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & x \end{pmatrix}$

3.5 $\begin{pmatrix} 2x & y & 0 & 0 \\ 2z & 0 & -2y & 0 \\ 0 & -z & -2x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\underline{dim\ \mathfrak{g} = 4}$

4.1 $\begin{pmatrix} x & z & 0 & t \\ 0 & y & -t & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -z & -y \end{pmatrix}$ 4.2 $\begin{pmatrix} x & y & 0 & 0 \\ z & t & 0 & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & -t \end{pmatrix}$

4.3 $\begin{pmatrix} x & y & 0 & t \\ z & -x & -t & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & x \end{pmatrix}$

<u>*dim* $\mathfrak{g} = 5$</u>

5.1 $$\begin{pmatrix} x & y & 0 & u \\ z & t & -u & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & -t \end{pmatrix}$$

<u>*dim* $\mathfrak{g} = 6$</u>

6.1 $$\begin{pmatrix} x & y & 0 & u \\ z & t & -u & 0 \\ 0 & v & -x & -z \\ -v & 0 & -y & -t \end{pmatrix}$$

Remark. *To simplify the computation, instead of $\mathfrak{so}(4,\mathbb{C})$ we use the linear Lie algebra 6.1, which is conjugate to $\mathfrak{so}(4,\mathbb{C})$.*

2. Classification of complex pairs

Preliminaries:

1. Let $\mathfrak{g}$ be one of the subalgebras of the Lie algebra $\mathfrak{so}(4,\mathbb{C})$ determined in Theorem 1. We assume that the Lie algebra $\mathfrak{g}$ acts naturally on $\mathbb{C}^4$; then $(\mathfrak{g},\mathbb{C}^4)$ is a faithful generalized module. The enumeration of the generalized modules obtained in this way coincide with that of the corresponding subalgebras of $\mathfrak{so}(4,\mathbb{C})$ in Theorem 1.

We say that a pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has type $n.m$, if the corresponding generalized module $(\mathfrak{g},\bar{\mathfrak{g}}/\mathfrak{g})$ is isomorphic to the generalized module $n.m$, i.e., to the generalized module $(\mathfrak{g},\mathbb{C}^4)$, where $\mathfrak{g}$ is the subalgebra of $\mathfrak{so}(4,\mathbb{C})$ supplied with the number $n.m$ in Theorem 1.

2. Let $(\bar{\mathfrak{g}},\mathfrak{g})$ be a pair of type $n.m$. Then without loss of generality we can identify the Lie algebra $\mathfrak{g}$ with the subalgebra $n.m$ of the Lie algebra $\mathfrak{so}(4,\mathbb{C})$.

Let $\{u_1,u_2,u_3,u_4\}$ be the standard basis of $\mathbb{C}^4$:

$$u_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad u_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad u_3=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad u_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

3. We define a pair $(\bar{\mathfrak{g}},\mathfrak{g})$ by the commutation table of the Lie algebra $\bar{\mathfrak{g}}$ only. Here by $\{e_1,\dots,e_n,u_1,u_2,u_3,u_4\}$ we denote a basis of $\bar{\mathfrak{g}}$ ($n=\dim\mathfrak{g}$). We assume that the Lie algebra $\mathfrak{g}$ is generated by $e_1,\dots,e_n$.

By p, r, s, etc. we denote the parameters appearing in the process of the classification. If there are some complementary conditions on them, it is indicated just after the table. Otherwise we assume that these parameters run through $\mathbb{C}$.

Theorem 2. *Any complex isotropically faithful pseudo-Riemannian pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of codimension 4 is equivalent to one and only one of the following:*

0.1 $\mathfrak{g} = \{0\}$

$\bar{\mathfrak{g}}_1$	u_1	u_2	u_3	u_4
u_1	0	u_3	u_2	0
u_2	$-u_3$	0	u_1	0
u_3	$-u_2$	$-u_1$	0	0
u_4	0	0	0	0

$\bar{\mathfrak{g}}_2$	u_1	u_2	u_3	u_4
u_1	0	u_3	0	u_1
u_2	$-u_3$	0	0	pu_2
u_3	0	0	0	$(p+1)u_3$
u_4	$-u_1$	$-pu_2$	$-(p+1)u_3$	0

$\bar{\mathfrak{g}}_3$	u_1	u_2	u_3	u_4
u_1	0	0	0	$2u_1$
u_2	0	0	u_1	u_2
u_3	0	$-u_1$	0	u_2+u_3
u_4	$-2u_1$	$-u_2$	$-u_2-u_3$	0

$\bar{\mathfrak{g}}_4$	u_1	u_2	u_3	u_4
u_1	0	0	0	u_1
u_2	0	0	0	u_1+u_2
u_3	0	0	0	u_2+u_3
u_4	$-u_1$	$-u_1-u_2$	$-u_2-u_3$	0

$\bar{\mathfrak{g}}_5$	u_1	u_2	u_3	u_4
u_1	0	0	0	u_1
u_2	0	0	0	u_1+u_2
u_3	0	0	0	pu_3
u_4	$-u_1$	$-u_1-u_2$	$-pu_3$	0

$\bar{\mathfrak{g}}_6$	u_1	u_2	u_3	u_4
u_1	0	0	0	u_1
u_2	0	0	0	pu_2
u_3	0	0	0	ru_3
u_4	$-u_1$	$-pu_2$	$-ru_3$	0

$\bar{\mathfrak{g}}_7$	u_1	u_2	u_3	u_4
u_1	0	0	u_1	0
u_2	0	0	0	u_2
u_3	$-u_1$	0	0	0
u_4	0	$-u_2$	0	0

$\bar{\mathfrak{g}}_8$	u_1	u_2	u_3	u_4
u_1	0	0	0	u_2
u_2	0	0	0	0
u_3	0	0	0	u_1
u_4	$-u_2$	0	$-u_1$	0

$\bar{\mathfrak{g}}_9$	u_1	u_2	u_3	u_4
u_1	0	0	0	u_1
u_2	0	0	0	0
u_3	0	0	0	u_2
u_4	$-u_1$	0	$-u_2$	0

$\bar{\mathfrak{g}}_{10}$	u_1	u_2	u_3	u_4
u_1	0	0	0	0
u_2	0	0	u_1	0
u_3	0	$-u_1$	0	0
u_4	0	0	0	0

$\bar{\mathfrak{g}}_{11}$	u_1	u_2	u_3	u_4
u_1	0	0	0	0
u_2	0	0	0	0
u_3	0	0	0	0
u_4	0	0	0	0

1.1

$$\mathfrak{g} = \left\{ \left.\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -\lambda x \end{pmatrix} \right| x \in \mathbb{C} \right\},$$

$$\lambda \in \mathbb{C},\ |\lambda| < 1,\ -\tfrac{\pi}{2} < \arg\lambda \leqslant \tfrac{\pi}{2} \text{ or } |\lambda| = 1,\ 0 \leqslant \arg\lambda \leqslant \tfrac{\pi}{2}$$

$\lambda = 0$

1.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	u_2	0
u_2	0	0	0	0	u_2
u_3	u_3	$-u_2$	0	0	u_3
u_4	0	0	$-u_2$	$-u_3$	0

2.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	pu_2
u_3	u_3	0	0	0	u_3
u_4	0	0	$-pu_2$	$-u_3$	0

3.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	e_1+u_2	0
u_2	0	0	0	0	0
u_3	u_3	$-e_1-u_2$	0	0	0
u_4	0	0	0	0	0

4.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	u_2	0
u_2	0	0	0	0	0
u_3	u_3	$-u_2$	0	0	0
u_4	0	0	0	0	0

5.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	e_1	0
u_2	0	0	0	0	u_2
u_3	u_3	$-e_1$	0	0	0
u_4	0	0	$-u_2$	0	0

6.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	u_2
u_3	u_3	0	0	0	0
u_4	0	0	$-u_2$	0	0

7.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	e_1	0
u_2	0	0	0	0	0
u_3	u_3	$-e_1$	0	0	0
u_4	0	0	0	0	0

$\lambda = \frac{1}{2}$

8.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	$\frac{1}{2}u_2$	$-u_3$	$-\frac{1}{2}u_4$
u_1	$-u_1$	0	0	$-2e_1$	u_2
u_2	$-\frac{1}{2}u_2$	0	0	u_4	0
u_3	u_3	$2e_1$	$-u_4$	0	0
u_4	$\frac{1}{2}u_4$	$-u_2$	0	0	0

9.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	$\frac{1}{2}u_2$	$-u_3$	$-\frac{1}{2}u_4$
u_1	$-u_1$	0	0	0	u_2
u_2	$-\frac{1}{2}u_2$	0	0	0	0
u_3	u_3	0	0	0	0
u_4	$\frac{1}{2}u_4$	$-u_2$	0	0	0

$|\lambda| < 1,\ -\frac{\pi}{2} < \arg\lambda \leqslant \frac{\pi}{2}$ or $|\lambda| = 1,\ 0 \leqslant \arg\lambda \leqslant \frac{\pi}{2}$

10.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	λu_2	$-u_3$	$-\lambda u_4$
u_1	$-u_1$	0	0	0	0
u_2	$-\lambda u_2$	0	0	0	0
u_3	u_3	0	0	0	0
u_4	λu_4	0	0	0	0

1.2

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & x & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -x & -x \end{pmatrix} \middle| \, x \in \mathbb{C} \right\}$$

1.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	u_1+u_2	$-u_3-u_4$	$-u_4$
u_1	$-u_1$	0	0	0	0
u_2	$-u_1-u_2$	0	0	0	0
u_3	u_3+u_4	0	0	0	0
u_4	u_4	0	0	0	0

1.3

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, x \in \mathbb{C} \right\}$$

1.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	e_1	0	u_1	u_2
u_1	$-e_1$	0	$-\frac{1}{2}u_2$	u_3	$\frac{1}{2}u_4$
u_2	0	$\frac{1}{2}u_2$	0	$\frac{1}{2}u_4$	0
u_3	$-u_1$	$-u_3$	$-\frac{1}{2}u_4$	0	0
u_4	$-u_2$	$-\frac{1}{2}u_4$	0	0	0

2.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-\lambda e_1+(\lambda+1)u_1+\lambda u_2$	0
u_2	0	0	0	0	u_2
u_3	$-u_1$	$\lambda e_1-(\lambda+1)u_1-\lambda u_2$	0	0	0
u_4	$-u_2$	0	$-u_2$	0	0 ,

$$|\lambda| < 1 \text{ or } |\lambda| = 1,\ 0 \leqslant \arg \lambda \leqslant \pi$$

3.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	u_1	0
u_2	0	0	0	0	u_2
u_3	$-u_1$	$-u_1$	0	0	e_1
u_4	$-u_2$	0	$-u_2$	$-e_1$	0

4.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	x	y
u_2	0	0	0	y	z
u_3	$-u_1$	$-x$	$-y$	0	0
u_4	$-u_2$	$-y$	$-z$	0	0,

where

$$x=\frac{1}{1+\lambda}e_1+\frac{\lambda}{1+\lambda}u_1-\frac{1}{1+\lambda}u_2,$$
$$y=-\frac{1}{1+\lambda}e_1+\frac{1}{1+\lambda}u_1+\frac{1}{1+\lambda}u_2,$$
$$z=-\frac{\lambda}{1+\lambda}e_1+\frac{\lambda}{1+\lambda}u_1+\frac{1+2\lambda}{1+\lambda}u_2,$$

$\lambda\neq-1$

5.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	u_1	u_2
u_3	$-u_1$	0	$-u_1$	0	$-u_3$
u_4	$-u_2$	0	$-u_2$	u_3	0

6.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	λu_1	$-\lambda e_1+(\lambda+1)u_2$
u_3	$-u_1$	0	$-\lambda u_1$	0	$-\lambda u_3$
u_4	$-u_2$	0	$\lambda e_1-(\lambda+1)u_2$	λu_3	0

7.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	u_2
u_3	$-u_1$	0	0	0	e_1
u_4	$-u_2$	0	$-u_2$	$-e_1$	0

8.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	$-u_1$	e_1
u_3	$-u_1$	0	u_1	0	e_1+u_3
u_4	$-u_2$	0	$-e_1$	$-e_1-u_3$	0

9.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	μu_1	$-\lambda\mu e_1+(\lambda+\mu)u_2$
u_3	$-u_1$	0	$-\mu u_1$	0	$(1-\mu)u_3$
u_4	$-u_2$	$-u_1$	$\lambda\mu e_1-(\lambda+\mu)u_2$	$(\mu-1)u_3$	0

10.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	$\frac{1}{2}u_1$	$-\frac{\lambda}{2}e_1+(\lambda+\frac{1}{2})u_2$
u_3	$-u_1$	0	$-\frac{1}{2}u_1$	0	$e_1+\frac{1}{2}u_3$
u_4	$-u_2$	$-u_1$	$\frac{\lambda}{2}e_1-(\lambda+\frac{1}{2})u_2$	$-e_1-\frac{1}{2}u_3$	0

11.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	$(1-\lambda)u_1$	$\lambda(\lambda-1)e_1+u_2$
u_3	$-u_1$	0	$(\lambda-1)u_1$	0	$e_1+\lambda u_3$
u_4	$-u_2$	$-u_1$	$\lambda(1-\lambda)e_1-u_2$	$-e_1-\lambda u_3$	0

$$\lambda \neq \tfrac{1}{2}$$

12.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-e_1+2u_1$	u_2
u_2	0	0	0	u_2	$-e_1+u_1$
u_3	$-u_1$	e_1-2u_1	$-u_2$	0	0
u_4	$-u_2$	$-u_2$	e_1-u_1	0	0

13.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	u_1
u_3	$-u_1$	0	0	0	e_1
u_4	$-u_2$	0	$-u_1$	$-e_1$	0

14.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	u_1
u_3	$-u_1$	0	0	0	0
u_4	$-u_2$	0	$-u_1$	0	0

15.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	u_1	$-e_1+u_1+2u_2$
u_3	$-u_1$	0	$-u_1$	0	0
u_4	$-u_2$	$-u_1$	$e_1-u_1-2u_2$	0	0

16.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	u_1	u_2-u_1
u_3	$-u_1$	0	$-u_1$	0	$-u_3$
u_4	$-u_2$	0	u_1-u_2	u_3	0

17.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	λu_1	$-\lambda e_1+(1-\lambda)u_1+(1+\lambda)u_2$
u_3	$-u_1$	0	$-\lambda u_1$	0	$(1-\lambda)u_3$
u_4	$-u_2$	$-u_1$	$\lambda e_1+(\lambda-1)u_1-(1+\lambda)u_2$	$(\lambda-1)u_3$	0

$$\lambda \neq 1$$

18.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	$\frac{1}{2}u_1$	$-\frac{1}{2}e_1+\frac{1}{2}u_1+\frac{3}{2}u_2$
u_3	$-u_1$	0	$-\frac{1}{2}u_1$	0	$e_1+\frac{1}{2}u_3$
u_4	$-u_2$	$-u_1$	$\frac{1}{2}e_1-\frac{1}{2}u_1-\frac{3}{2}u_2$	$-e_1-\frac{1}{2}u_3$	0

19.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	0	u_1+u_2
u_3	$-u_1$	0	0	0	e_1+u_3
u_4	$-u_2$	$-u_1$	$-u_1-u_2$	$-e_1-u_3$	0

20.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$(1-2\lambda)e_1+2\lambda u_1$	$(2\lambda-1)u_2$
u_2	0	0	0	λu_2	$\frac{2\lambda-1}{2(\lambda-1)}e_1-\frac{1}{2(\lambda-1)}u_1$
u_3	$-u_1$	$(2\lambda-1)e_1-2\lambda u_1$	$-\lambda u_2$	0	$(\lambda-1)u_4$
u_4	$-u_2$	$(1-2\lambda)u_2$	$\frac{1-2\lambda}{2(\lambda-1)}e_1+\frac{1}{2(\lambda-1)}u_1$	$(1-\lambda)u_4$	0

$$\lambda \neq 1$$

21.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-\frac{1}{3}e_1+\frac{4}{3}u_1$	$\frac{1}{3}u_2$
u_2	0	0	0	$\frac{2}{3}u_2$	$-\frac{1}{2}e_1+\frac{3}{2}u_1$
u_3	$-u_1$	$\frac{1}{3}e_1-\frac{4}{3}u_1$	$-\frac{2}{3}u_2$	0	$e_1-\frac{1}{3}u_4$
u_4	$-u_2$	$-\frac{1}{3}u_2$	$\frac{1}{2}e_1-\frac{3}{2}u_1$	$\frac{1}{3}u_4-e_1$	0

22.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$2u_1$	$2u_2$
u_2	0	0	0	u_2	$e_1-\frac{1}{2}u_1$
u_3	$-u_1$	$-2u_1$	$-u_2$	0	u_4
u_4	$-u_2$	$-2u_2$	$\frac{1}{2}u_1-e_1$	$-u_4$	0

23.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	x	y
u_2	0	0	0	y	z
u_3	$-u_1$	$-x$	$-y$	0	0
u_4	$-u_2$	$-y$	$-z$	0	0,

where

$$x = \frac{\lambda\mu(\lambda-1)}{\lambda+\mu-\lambda\mu}e_1 + \frac{\lambda^2+\mu-\lambda^2\mu}{\lambda+\mu-\lambda\mu}u_1 + \frac{\lambda(1-\lambda)}{\lambda+\mu-\lambda\mu}u_2,$$
$$y = -\frac{\lambda\mu}{\lambda+\mu-\lambda\mu}e_1 + \frac{\mu}{\lambda+\mu-\lambda\mu}u_1 + \frac{\lambda}{\lambda+\mu-\lambda\mu}u_2,$$
$$z = \frac{\lambda\mu(\mu-1)}{\lambda+\mu-\lambda\mu}e_1 + \frac{\mu(1-\mu)}{\lambda+\mu-\lambda\mu}u_1 + \frac{\lambda+\mu^2-\mu^2\lambda}{\lambda+\mu-\lambda\mu}u_2,$$

$$\lambda+\mu-\lambda\mu \neq 0.$$

Two pairs corresponding to parameters (λ_1,μ_1) and (λ_2,μ_2) are equivalent if and only if the points $(\lambda_1,\mu_1),\ (\lambda_2,\mu_2) \in \mathbb{C}^* \times \mathbb{C}^*$ lie in the same orbit of the action of the symmetric group $\mathfrak{S}_3$ on $\mathbb{C}^* \times \mathbb{C}^*$ generated by the transformations

$$(\lambda,\mu) \to (\mu,\lambda); \qquad (\lambda,\mu) \to (\frac{1}{\lambda}, -\frac{\mu}{\lambda}).$$

24.

[,]	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	0
u_3	$-u_1$	0	0	0	e_1
u_4	$-u_2$	0	0	$-e_1$	0

25.

[,]	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	0
u_3	$-u_1$	0	0	0	0
u_4	$-u_2$	0	0	0	0

1.4

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| x \in \mathbb{C} \right\}$$

1.

[,]	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	e_1
u_1	0	0	u_1	u_2	u_1
u_2	$-u_1$	$-u_1$	0	u_3	0
u_3	$-u_2$	$-u_2$	$-u_3$	0	$-u_3$
u_4	$-e_1$	$-u_1$	0	u_3	0

2.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	e_1
u_1	0	0	0	0	pu_1
u_2	$-u_1$	0	0	0	$(p-1)u_2$
u_3	$-u_2$	0	0	0	$(p-2)u_3$
u_4	$-e_1$	$-pu_1$	$(1-p)u_2$	$(2-p)u_3$	0

3.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	e_1
u_1	0	0	0	0	$2u_1$
u_2	$-u_1$	0	0	e_1	u_2
u_3	$-u_2$	0	$-e_1$	0	0
u_4	$-e_1$	$-2u_1$	$-u_2$	0	0

4.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	u_1	u_2	0
u_2	$-u_1$	$-u_1$	0	u_3	0
u_3	$-u_2$	$-u_2$	$-u_3$	0	0
u_4	0	0	0	0	0

5.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	u_1
u_2	$-u_1$	0	0	0	u_2
u_3	$-u_2$	0	0	0	u_1+u_3
u_4	0	$-u_1$	$-u_2$	$-u_1-u_3$	0

6.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	u_1
u_2	$-u_1$	0	0	0	u_2
u_3	$-u_2$	0	0	0	u_3
u_4	0	$-u_1$	$-u_2$	$-u_3$	0

7.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	$re_1+u_2+u_4$	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2-u_4$	0	pu_4
u_4	0	0	0	$-pu_4$	0

8.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	re_1+u_2	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2$	0	pu_4
u_4	0	0	0	$-pu_4$	0

9.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	$re_1+u_2+u_4$	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2-u_4$	0	u_1-u_4
u_4	0	0	0	u_4-u_1	0

10.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	re_1+u_2	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2$	0	u_1-u_4
u_4	0	0	0	u_4-u_1	0

11.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	re_1+u_4	0
u_3	$-u_2$	0	$-re_1-u_4$	0	u_4
u_4	0	0	0	$-u_4$	0

12.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	re_1	0
u_3	$-u_2$	0	$-re_1$	0	u_4
u_4	0	0	0	$-u_4$	0

13.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1+u_4	0
u_3	$-u_2$	0	$-e_1-u_4$	0	u_1
u_4	0	0	0	$-u_1$	0

14.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	u_4	0
u_3	$-u_2$	0	$-u_4$	0	u_1
u_4	0	0	0	$-u_1$	0

15.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1+u_4	0
u_3	$-u_2$	0	$-e_1-u_4$	0	0
u_4	0	0	0	0	0

16.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	u_4	0
u_3	$-u_2$	0	$-u_4$	0	0
u_4	0	0	0	0	0

17.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1	0
u_3	$-u_2$	0	$-e_1$	0	u_1
u_4	0	0	0	$-u_1$	0

18.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	0
u_3	$-u_2$	0	0	0	u_1
u_4	0	0	0	$-u_1$	0

19.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1	0
u_3	$-u_2$	0	$-e_1$	0	0
u_4	0	0	0	0	0

20.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	0
u_3	$-u_2$	0	0	0	0
u_4	0	0	0	0	0

2.1

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -y \end{pmatrix} \,\middle|\, x, y \in \mathbb{C} \right\}$$

1.

$[,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_2$	0	0	0	e_2
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_4	0	$-e_2$	0	0

2.

$[,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_2$	0	0	0	0
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_4	0	0	0	0

3.

$[,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	0
u_3	u_3	0	0	0	0	0
u_4	0	u_4	0	0	0	0

2.2

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & y & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & -\lambda x \end{pmatrix} \right| x, y \in \mathbb{C} \right\},$$

$$\lambda \in \mathbb{C},\ |\lambda| < 1 \text{ or } |\lambda| = 1,\ 0 \leqslant \arg\lambda \leqslant \pi$$

$\lambda = 0$

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	$-u_4$	$-2e_2$
u_1	$-u_1$	0	0	0	u_2	$-u_1$
u_2	0	$-u_1$	0	0	0	u_2
u_3	u_3	u_4	$-u_2$	0	0	$2u_3$
u_4	0	$2e_2$	u_1	$-u_2$	$-2u_3$	0

2.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	e_2	u_4	0
u_2	0	$-u_1$	$-e_2$	0	$(p-1)u_3$	pu_4
u_3	u_3	u_4	$-u_4$	$(1-p)u_3$	0	0
u_4	0	0	0	$-pu_4$	0	0

3.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_1$	0	0	u_3	u_4
u_3	u_3	u_4	0	$-u_3$	0	0
u_4	0	0	0	$-u_4$	0	0

$\lambda = 1$

4.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	e_2	0
u_2	$-u_2$	$-u_1$	0	0	e_1	e_2
u_3	u_3	u_4	$-e_2$	$-e_1$	0	0
u_4	u_4	0	0	$-e_2$	0	0

5.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	e_2	0
u_3	u_3	u_4	0	$-e_2$	0	0
u_4	u_4	0	0	0	0	0

$\lambda = -\frac{1}{2}$

6.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$\frac{3}{2}e_2$	u_1	$-\frac{1}{2}u_2$	$-u_3$	$\frac{1}{2}u_4$
e_2	$-\frac{3}{2}e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	u_4	0	0
u_2	$\frac{1}{2}u_2$	$-u_1$	$-u_4$	0	0	0
u_3	u_3	u_4	0	0	0	0
u_4	$-\frac{1}{2}u_4$	0	0	0	0	0

$|\lambda| < 1$ or $|\lambda| = 1,\ 0 \leqslant \arg\lambda \leqslant \pi$

7.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$(1-\lambda)e_2$	u_1	λu_2	$-u_3$	$-\lambda u_4$
e_2	$(\lambda-1)e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_2$	$-u_1$	0	0	0	0
u_3	u_3	u_4	0	0	0	0
u_4	λu_4	0	0	0	0	0

2.3

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & y & 0 & x \\ 0 & -x & -x & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & x \end{pmatrix} \right| x, y \in \mathbb{C} \right\}$$

1.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	u_1	$-u_2$	$-u_2-u_3$	u_1+u_4
e_2	$-2e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0
u_3	u_2+u_3	u_4	0	0	0	0
u_4	$-u_1-u_4$	0	0	0	0	0

2.4

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \; x, y \in \mathbb{C} \right\}$$

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	u_2	0
u_1	$-u_1$	0	0	u_1	u_2	0
u_2	0	$-u_1$	$-u_1$	0	u_3	0
u_3	u_3	$-u_2$	$-u_2$	$-u_3$	0	0
u_4	0	0	0	0	0	0

2.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	u_2	0
u_1	$-u_1$	0	0	0	0	u_1
u_2	0	$-u_1$	0	0	0	u_2
u_3	u_3	$-u_2$	0	0	0	u_3
u_4	0	0	$-u_1$	$-u_2$	$-u_3$	0

3.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	u_2	0
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_1$	0	0	0	0
u_3	u_3	$-u_2$	0	0	0	0
u_4	0	0	0	0	0	0

2.5

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & x & 0 & y \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \end{pmatrix} \middle| \; x, y \in \mathbb{C} \right\}$$

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	$-2e_1$
e_2	0	0	0	$-2e_2$	$-u_2$	u_1
u_1	0	0	0	$2e_2-u_1$	u_2+u_4	$2e_1-u_1$
u_2	$-u_1$	$2e_2$	u_1-2e_2	0	$-2u_3$	u_2-u_4
u_3	u_4	u_2	$-u_2-u_4$	$2u_3$	0	$2u_3$
u_4	$2e_1$	$-u_1$	u_1-2e_1	u_4-u_2	$-2u_3$	0

2.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	$-2e_2$	$-u_2$	u_1
u_1	0	0	0	$-u_1$	u_4	0
u_2	$-u_1$	$2e_2$	u_1	0	$-2u_3$	$-u_4$
u_3	u_4	u_2	$-u_4$	$2u_3$	0	0
u_4	0	$-u_1$	0	u_4	0	0

3.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	u_1	0
u_2	$-u_1$	0	0	0	$e_1+ge_2+(1-h)u_2$	hu_1
u_3	u_4	u_2	$-u_1$	$-e_1-ge_2+(h-1)u_2$	0	$-(g+h)e_1+ke_2-(1+h)u_4$
u_4	0	$-u_1$	0	$-hu_1$	$(g+h)e_1-ke_2+(1+h)u_4$	0

$$\operatorname{Re} h > 0 \text{ or } \operatorname{Re} h = 0,\ \operatorname{Im} h \geqslant 0 \text{ (if } k \neq 0),\ h \in \mathbb{C} \text{ (if } k = 0)$$

4.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	u_1	0
u_2	$-u_1$	0	0	0	$ge_2+(1-h)u_2$	hu_1
u_3	u_4	u_2	$-u_1$	$-ge_2+(h-1)u_2$	0	$-(g+h)e_1-(1+h)u_4$
u_4	0	$-u_1$	0	$-hu_1$	$(g+h)e_1+(1+h)u_4$	0

$$\operatorname{Re} h > 0 \text{ or } \operatorname{Re} h = 0,\ \operatorname{Im} h \geqslant 0$$

5.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	$e_1+ge_2-u_2$	u_1
u_3	u_4	u_2	0	$-e_1-ge_2+u_2$	0	$-ge_1+ke_2-u_4$
u_4	0	$-u_1$	0	$-u_1$	$ge_1-ke_2+u_4$	0

6.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	ge_2-u_2	u_1
u_3	u_4	u_2	0	$-ge_2+u_2$	0	$-ge_1-u_4$
u_4	0	$-u_1$	0	$-u_1$	ge_1+u_4	0

7.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1+e_2	0
u_3	u_4	u_2	0	$-e_1-e_2$	0	$-e_1+ke_2$
u_4	0	$-u_1$	0	0	e_1-ke_2	0

8.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_2	0
u_3	u_4	u_2	0	$-e_2$	0	$-e_1$
u_4	0	$-u_1$	0	0	e_1	0

9.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1	0
u_3	u_4	u_2	0	$-e_1$	0	e_2
u_4	0	$-u_1$	0	0	$-e_2$	0

10.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1	0
u_3	u_4	u_2	0	$-e_1$	0	0
u_4	0	$-u_1$	0	0	0	0

11.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	0	0
u_3	u_4	u_2	0	0	0	0
u_4	0	$-u_1$	0	0	0	0

3.1

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & z & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -z & -y \end{pmatrix} \middle| \; x, y, z \in \mathbb{C} \right\}$$

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	0	e_3	u_1	0	$-u_3$	0
e_2	0	0	$-e_3$	0	u_2	0	$-u_4$
e_3	$-e_3$	e_3	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0	0
u_2	0	$-u_2$	$-u_1$	0	0	0	0
u_3	u_3	0	u_4	0	0	0	0
u_4	0	u_4	0	0	0	0	0

3.2

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & y & 0 & z \\ 0 & \lambda x & -z & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & -\lambda x \end{pmatrix} \middle| \; x, y, z \in \mathbb{C} \right\},$$

$$\lambda \in \mathbb{C},\ \operatorname{Re}\lambda > 0 \text{ or } \operatorname{Re}\lambda = 0,\ \operatorname{Im}\lambda \geqslant 0$$

$\lambda = 0$

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	e_2	e_3	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	0	u_1	$-u_4$	$-2e_2$
e_3	$-e_3$	0	0	0	$-2e_3$	$-u_2$	u_1
u_1	$-u_1$	0	0	0	$2e_3-u_1$	u_2+u_4	$2e_2-u_1$
u_2	0	$-u_1$	$2e_3$	u_1-2e_3	0	$-2u_3$	u_2-u_4
u_3	u_3	u_4	u_2	$-u_2-u_4$	$2u_3$	0	$2u_3$
u_4	0	$2e_2$	$-u_1$	u_1-2e_2	u_4-u_2	$-2u_3$	0

2.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	e_2	e_3	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	0	u_1	$-u_4$	$-2e_2$
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	u_2	$-u_1$
u_2	0	$-u_1$	0	0	0	0	u_2
u_3	u_3	u_4	u_2	$-u_2$	0	0	$2u_3$
u_4	0	$2e_2$	$-u_1$	u_1	$-u_2$	$-2u_3$	0

$\lambda = 1$

3.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	0	u_1	$-u_4$	0
e_3	$-2e_3$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	e_2	0
u_3	u_3	u_4	u_2	0	$-e_2$	0	0
u_4	u_4	0	$-u_1$	0	0	0	0

$\operatorname{Re}\lambda > 0$ or $\operatorname{Re}\lambda = 0,\ \operatorname{Im}\lambda \geqslant 0$

4.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$(1-\lambda)e_2$	$(1+\lambda)e_3$	u_1	λu_2	$-u_3$	$-\lambda u_4$
e_2	$(\lambda-1)e_2$	0	0	0	u_1	$-u_4$	0
e_3	$-(1+\lambda)e_3$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	0	0
u_2	$-\lambda u_2$	$-u_1$	0	0	0	0	0
u_3	u_3	u_4	u_2	0	0	0	0
u_4	λu_4	0	$-u_1$	0	0	0	0

3.3

$$\mathfrak{g} = \left\{ \left(\begin{array}{cccc} 0 & y & 0 & z \\ 0 & x & -z & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -y & -x \end{array} \right) \middle| \ x, y, z \in \mathbb{C} \right\}$$

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	u_1	0
u_2	$-u_2$	$-u_1$	0	0	0	pe_3+u_2	0
u_3	0	u_4	u_2	$-u_1$	$-pe_3-u_2$	0	$-pe_2-u_4$
u_4	u_4	0	$-u_1$	0	0	pe_2+u_4	0

2.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	e_3	0
u_3	0	u_4	u_2	0	$-e_3$	0	$-e_2$
u_4	u_4	0	$-u_1$	0	0	e_2	0

3.

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	0	0
u_3	0	u_4	u_2	0	0	0	0
u_4	u_4	0	$-u_1$	0	0	0	0

3.4

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & y & 0 & 0 \\ z & -x & 0 & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & x \end{pmatrix} \middle| \; x, y, z \in \mathbb{C} \right\}$$

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	u_1	$-u_2$	$-u_3$	u_4
e_2	$-2e_2$	0	e_1	0	u_1	$-u_4$	0
e_3	$2e_3$	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	0	$-u_2$	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0	0
u_3	u_3	u_4	0	0	0	0	0
u_4	$-u_4$	0	u_3	0	0	0	0

3.5

$$\mathfrak{g} = \left\{ \begin{pmatrix} 2x & y & 0 & 0 \\ 2z & 0 & -2y & 0 \\ 0 & -z & -2x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \; x, y, z \in \mathbb{C} \right\}$$

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	0	0	u_1
u_2	0	$-u_1$	u_3	0	0	0	u_2
u_3	$2u_3$	$2u_2$	0	0	0	0	u_3
u_4	0	0	0	$-u_1$	$-u_2$	$-u_3$	0

2.

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	e_2	e_1	0
u_2	0	$-u_1$	u_3	$-e_2$	0	e_3	0
u_3	$2u_3$	$2u_2$	0	$-e_1$	$-e_3$	0	0
u_4	0	0	0	0	0	0	0

3.

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	0	0	0
u_2	0	$-u_1$	u_3	0	0	0	0
u_3	$2u_3$	$2u_2$	0	0	0	0	0
u_4	0	0	0	0	0	0	0

4.1

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & z & 0 & t \\ 0 & y & -t & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -z & -y \end{pmatrix} \right| x, y, z, t \in \mathbb{C} \right\}$$

1.

[,]	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	e_3	e_4	u_1	0	$-u_3$	0
e_2	0	0	$-e_3$	e_4	0	u_2	0	$-u_4$
e_3	$-e_3$	e_3	0	0	0	u_1	$-u_4$	0
e_4	$-e_4$	$-e_4$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	0	0	0
u_2	0	$-u_2$	$-u_1$	0	0	0	0	0
u_3	u_3	0	u_4	u_2	0	0	0	0
u_4	0	u_4	0	$-u_1$	0	0	0	0

4.2

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & y & 0 & 0 \\ z & t & 0 & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & -t \end{pmatrix} \right| x, y, z, t \in \mathbb{C} \right\}$$

1.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	0	e_1+3e_2	$2e_3$
u_2	u_2	$-u_2$	$-u_1$	0	0	0	$2e_4$	$-e_1+3e_2$
u_3	u_3	u_3	u_4	0	$-e_1-3e_2$	$-2e_4$	0	0
u_4	$-u_4$	u_4	0	u_3	$-2e_3$	e_1-3e_2	0	0

2.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	0	0	0
u_2	u_2	$-u_2$	$-u_1$	0	0	0	0	0
u_3	u_3	u_3	u_4	0	0	0	0	0
u_4	$-u_4$	u_4	0	u_3	0	0	0	0

4.3

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & y & 0 & t \\ z & -x & -t & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & x \end{pmatrix} \right| x, y, z, t \in \mathbb{C} \right\}$$

1.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	0	u_1	$-u_2$	$-u_3$	u_4
e_2	$-2e_2$	0	e_1	0	0	u_1	$-u_4$	0
e_3	$2e_3$	$-e_1$	0	0	u_2	0	0	$-u_3$
e_4	0	0	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	$-u_2$	0	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0	0	0
u_3	u_3	u_4	0	u_2	0	0	0	e_4
u_4	$-u_4$	0	u_3	$-u_1$	0	0	$-e_4$	0

2.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	0	u_1	$-u_2$	$-u_3$	u_4
e_2	$-2e_2$	0	e_1	0	0	u_1	$-u_4$	0
e_3	$2e_3$	$-e_1$	0	0	u_2	0	0	$-u_3$
e_4	0	0	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	$-u_2$	0	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0	0	0
u_3	u_3	u_4	0	u_2	0	0	0	0
u_4	$-u_4$	0	u_3	$-u_1$	0	0	0	0

5.1

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & y & 0 & u \\ z & t & -u & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & -t \end{pmatrix} \right| x, y, z, t, u \in \mathbb{C} \right\}$$

1.

$[\,,]$	e_1	e_2	e_3	e_4	e_5	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	0	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	$2e_5$	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	0	u_2	0	0	$-u_3$
e_5	0	$-2e_5$	0	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	0	0	0	0
u_2	u_2	$-u_2$	$-u_1$	0	0	0	0	0	0
u_3	u_3	u_3	u_4	0	u_2	0	0	0	0
u_4	$-u_4$	u_4	0	u_3	$-u_1$	0	0	0	0

6.1

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & y & 0 & u \\ z & t & -u & 0 \\ 0 & v & -x & -z \\ -v & 0 & -y & -t \end{pmatrix} \right| x, y, z, t, u, v \in \mathbb{C} \right\}$$

1.

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	0	0	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	$2e_5$	$-2e_6$	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	0	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	0	0	u_2	0	0	$-u_3$
e_5	0	$-2e_5$	0	0	0	$-e_2$	0	0	$-u_2$	u_1
e_6	0	$2e_6$	0	0	e_2	0	$-u_4$	u_3	0	0
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	u_4	0	$2e_5$	e_1+e_2	$2e_3$
u_2	u_2	$-u_2$	$-u_1$	0	0	$-u_3$	$-2e_5$	0	$2e_4$	$-e_1+e_2$
u_3	u_3	u_3	u_4	0	u_2	0	$-e_1-e_2$	$-2e_4$	0	$2e_6$
u_4	$-u_4$	u_4	0	u_3	$-u_1$	0	$-2e_3$	e_1-e_2	$-2e_6$	0

2.

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	0	0	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	$2e_5$	$-2e_6$	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	0	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	0	0	u_2	0	0	$-u_3$
e_5	0	$-2e_5$	0	0	0	$-e_2$	0	0	$-u_2$	u_1
e_6	0	$2e_6$	0	0	e_2	0	$-u_4$	u_3	0	0
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	u_4	0	0	0	0
u_2	u_2	$-u_2$	$-u_1$	0	0	$-u_3$	0	0	0	0
u_3	u_3	u_3	u_4	0	u_2	0	0	0	0	0
u_4	$-u_4$	u_4	0	u_3	$-u_1$	0	0	0	0	0

CHAPTER II

CLASSIFICATION OF REAL PAIRS

PRELIMINARIES.

1. Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a complex isotropically faithful pseudo-Riemannian pair. An anti–involution σ of $\bar{\mathfrak{g}}$ is said to be an anti–involution of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ if $\sigma(\mathfrak{g}) = \mathfrak{g}$.

Every real form of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ coincides with the set of fixed points $(\bar{\mathfrak{g}}^{\sigma}, \mathfrak{g}^{\sigma})$ of some anti–involution σ of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$.

Two real forms $(\bar{\mathfrak{g}}^{\sigma_1}, \mathfrak{g}^{\sigma_1})$ and $(\bar{\mathfrak{g}}^{\sigma_2}, \mathfrak{g}^{\sigma_2})$ are equivalent if and only if the corresponding anti–involutions σ_1 and σ_2 are conjugate, i.e. if there exists an automorphism f of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ such that $\sigma_2 = \mathrm{f}\,\sigma_1\,\mathrm{f}^{-1}$.

Let $\mathcal{E} = \{e_1, .., e_n, u_1, .., u_4\}$ be a basis of $\bar{\mathfrak{g}}$, and let A, I_1, I_2 be the matrices of f, σ_1, σ_2 with respect to this basis. Then the condition that $(\bar{\mathfrak{g}}^{\sigma_1}, \mathfrak{g}^{\sigma_1})$ and $(\bar{\mathfrak{g}}^{\sigma_2}, \mathfrak{g}^{\sigma_2})$ are equivalent has the form $I_2 = AI_1\overline{A}^{-1}$.

2. Let m.n be a subalgebra of $\mathfrak{so}(4, \mathbb{C})$. By $\mathrm{m.n}^{\mathrm{k}}$ denote the real forms of m.n. The real pairs corresponding to the linear Lie algebra $\mathrm{m.n}^{\mathrm{k}}$ will be called the pairs of type $\mathrm{m.n}^{\mathrm{k}}$.

3. For each pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ we fix a basis $\mathcal{E} = \{e_1, .., e_n, u_1, .., u_4\}$, where $n = \dim \mathfrak{g}$. Then we can identify the group of automorphisms of $(\bar{\mathfrak{g}}, \mathfrak{g})$ with $\mathcal{A}$, where $\mathcal{A}$ is a subgroup of $\mathrm{GL}(n+4, \mathbb{C})$ such that each element of $\mathcal{A}$ is the matrix of some automorphism of $(\bar{\mathfrak{g}}, \mathfrak{g})$ with respect to the basis $\mathcal{E}$.

Similarly, we can identify the set of anti–involutions of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ with $\mathcal{I}$, where $\mathcal{I}$ is the set of matrices of anti–involutions of $(\bar{\mathfrak{g}}, \mathfrak{g})$ with respect to the basis $\mathcal{E}$.

Further, the group $\mathcal{A}$ will be called the group of automorphisms of $(\bar{\mathfrak{g}}, \mathfrak{g})$ and the set $\mathcal{I}$ the set of anti–involutions of $(\bar{\mathfrak{g}}, \mathfrak{g})$.

4. Let $v \in \bar{\mathfrak{g}}$. By X denote the row of coordinates of the vector v with respect to the basis $\mathcal{E}$. The expression $X = (a_1, .., a_n, x, y, z, t)$ means that

$$v = a_1e_1 + ... + a_ne_n + xu_1 + yu_2 + zu_3 + tu_4.$$

5. If the anti–involution of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equal to $E_n + 4$, then the real form of $(\bar{\mathfrak{g}}, \mathfrak{g})$ corresponding to the anti–involution E has the same structure constants as $(\bar{\mathfrak{g}}, \mathfrak{g})$.

Indeed, find the set of fixed points of the anti–involution E.

Let $v \in \bar{\mathfrak{g}}$. $E(X) = \overline{X} = X \Rightarrow X \in \mathbb{R}^{n+4}$.

Then we can choose the following basis of the set of fixed points $\mathcal{E}' = \mathcal{E} = \{e_1, .., e_n, u_1, .., u_4\}$. It follows that the multiplication table of the real form corresponding to the anti–involution E coincides with the multiplication table of $(\bar{\mathfrak{g}}, \mathfrak{g})$.

6. Let the linear Lie algebra $\mathfrak{g}$ have only one (up to the operation $AI\overline{A}^{-1}$) anti–involution: E.

Then there exists a basis of $\mathfrak{g}$ which consists of real elements and which is a basis of the real form at the same time (that is, ronghly speaking, the real form has the same form as the algebra).

The method of finding real pairs with faithful isotropic representation is similar to that of finding complex pairs (for a detailed description see [KT] or [K1]).

Note that since $\mathbb{R}$ is not algebraically closed, pairs which are equivalent over $\mathbb{C}$ can be nonequivalent over $\mathbb{R}$.

7. In the case $\mathfrak{g}^\sigma = \{0\}$ the classification of real isotropically faithful pairs $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ is equivalent to the classification (up to isomorphism) of all four–dimensional real Lie algebras $\bar{\mathfrak{g}}^\sigma$. It can be found, for example, in [L3].

Theorem 0.1. *Any isotropically faithful pair* $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 0.1 *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	u_1	u_2	u_3	u_4
u_1	0	u_3	u_2	0
u_2	$-u_3$	0	u_1	0
u_3	$-u_2$	$-u_1$	0	0
u_4	0	0	0	0

2.

$[\,,]$	u_1	u_2	u_3	u_4
u_1	0	$2u_2$	$-2u_3$	0
u_2	$-2u_2$	0	u_1	0
u_3	$2u_2$	$-u_1$	0	0
u_4	0	0	0	0

3.

$[\,,]$	u_1	u_2	u_3	u_4
u_1	0	u_3	0	u_1
u_2	$-u_3$	0	0	pu_2
u_3	0	0	0	$(p+1)u_3$
u_4	$-u_1$	$-pu_2$	$-(p+1)u_3$	0

4.

$[\,,]$	u_1	u_2	u_3	u_4
u_1	0	u_3	0	$pu_1 + u_2$
u_2	$-u_3$	0	0	$pu_2 - u_1$
u_3	0	0	0	$2pu_3$
u_4	$-pu_1 - u_2$	$u_1 - pu_2$	$-2pu_3$	0

$$p \geqslant 0$$

5.

$[\,,]$	u_1	u_2	u_3	u_4
u_1	0	0	0	$2u_1$
u_2	0	0	u_1	u_2
u_3	0	$-u_1$	0	u_2+u_3
u_4	$-2u_1$	$-u_2$	$-u_2-u_3$	0

6.

$[,]$	u_1	u_2	u_3	u_4
u_1	0	0	0	u_1
u_2	0	0	0	u_1+u_2
u_3	0	0	0	u_2+u_3
u_4	$-u_1$	$-u_1-u_2$	$-u_2-u_3$	0

7.

$[,]$	u_1	u_2	u_3	u_4
u_1	0	0	0	u_1
u_2	0	0	0	u_1+u_2
u_3	0	0	0	pu_3
u_4	$-u_1$	$-u_1-u_2$	$-pu_3$	0

8.

$[,]$	u_1	u_2	u_3	u_4
u_1	0	0	0	u_1
u_2	0	0	0	pu_2
u_3	0	0	0	ru_3
u_4	$-u_1$	$-pu_2$	$-ru_3$	0

9.

$[,]$	u_1	u_2	u_3	u_4
u_1	0	0	0	pu_1+u_2
u_2	0	0	0	pu_2-u_1
u_3	0	0	0	ru_3
u_4	$-pu_1-u_2$	u_1-pu_2	$-ru_3$	0

$$p \geqslant 0$$

10.

$[,]$	u_1	u_2	u_3	u_4
u_1	0	0	u_1	0
u_2	0	0	0	u_2
u_3	$-u_1$	0	0	0
u_4	0	$-u_2$	0	0

11.

$[,]$	u_1	u_2	u_3	u_4
u_1	0	0	u_1	$-u_2$
u_2	0	0	u_2	u_1
u_3	$-u_1$	$-u_2$	0	0
u_4	u_2	$-u_1$	0	0

12.

$[\,,]$	u_1	u_2	u_3	u_4
u_1	0	0	0	u_2
u_2	0	0	0	0
u_3	0	0	0	u_1
u_4	$-u_2$	0	$-u_1$	0

13.

$[\,,]$	u_1	u_2	u_3	u_4
u_1	0	0	0	u_1
u_2	0	0	0	0
u_3	0	0	0	u_2
u_4	$-u_1$	0	$-u_2$	0

14.

$[\,,]$	u_1	u_2	u_3	u_4
u_1	0	0	0	0
u_2	0	0	u_1	0
u_3	0	$-u_1$	0	0
u_4	0	0	0	0

15.

$[\,,]$	u_1	u_2	u_3	u_4
u_1	0	0	0	0
u_2	0	0	0	0
u_3	0	0	0	0
u_4	0	0	0	0

1. Real forms of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ with solvable subalgebra $\mathfrak{g}$ One-dimensional case

Theorem 1.1. *Any real form of the linear Lie algebra 1.1 is conjugate to one and only one of the following linear Lie algebras:*

$$1.1^1 \quad \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -\lambda x \end{pmatrix}, \lambda \in [0,1] \quad 1.1^2 \quad \begin{pmatrix} 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -\lambda x \\ x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \end{pmatrix}, \lambda \in [0,1]$$

$$1.1^3 \quad \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda x \\ 0 & 0 & -x & 0 \\ 0 & \lambda x & 0 & 0 \end{pmatrix}, \lambda \in]0,1] \quad 1.1^4 \quad \begin{pmatrix} 0 & 0 & -x & 0 \\ 0 & -\lambda x & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda x \end{pmatrix}, \lambda \in]0,1[$$

$$1.1^5 \quad \begin{pmatrix} x\cos\frac{\phi}{2} & x\sin\frac{\phi}{2} & 0 & 0 \\ -x\sin\frac{\phi}{2} & x\cos\frac{\phi}{2} & 0 & 0 \\ 0 & 0 & -x\cos\frac{\phi}{2} & -x\sin\frac{\phi}{2} \\ 0 & 0 & x\sin\frac{\phi}{2} & -x\cos\frac{\phi}{2} \end{pmatrix}, \ \phi \in]0, \tfrac{\pi}{2}]$$

$$1.1^6 \quad \begin{pmatrix} -x\sin\frac{\phi}{2} & x\cos\frac{\phi}{2} & 0 & 0 \\ -x\cos\frac{\phi}{2} & -x\sin\frac{\phi}{2} & 0 & 0 \\ 0 & 0 & x\sin\frac{\phi}{2} & x\cos\frac{\phi}{2} \\ 0 & 0 & -x\cos\frac{\phi}{2} & x\sin\frac{\phi}{2} \end{pmatrix}, \ \phi \in]0, \tfrac{\pi}{2}[$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ of type 1.1^1 is equivalent to one and only one of the following pairs:

$\lambda = 0$

1.

[,]	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	u_2	0
u_2	0	0	0	0	u_2
u_3	u_3	$-u_2$	0	0	u_3
u_4	0	0	$-u_2$	$-u_3$	0

2.

[,]	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	pu_2
u_3	u_3	0	0	0	u_3
u_4	0	0	$-pu_2$	$-u_3$	0

3.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	e_1+u_2	0
u_2	0	0	0	0	0
u_3	u_3	$-e_1-u_2$	0	0	0
u_4	0	0	0	0	0

4.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	u_2	0
u_2	0	0	0	0	0
u_3	u_3	$-u_2$	0	0	0
u_4	0	0	0	0	0

5.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	e_1	0
u_2	0	0	0	0	u_2
u_3	u_3	$-e_1$	0	0	0
u_4	0	0	$-u_2$	0	0

6.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	u_2
u_3	u_3	0	0	0	0
u_4	0	0	$-u_2$	0	0

7.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	e_1	0
u_2	0	0	0	0	0
u_3	u_3	$-e_1$	0	0	0
u_4	0	0	0	0	0

$\lambda = \frac{1}{2}$

8.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	$\frac{1}{2}u_2$	$-u_3$	$-\frac{1}{2}u_4$
u_1	$-u_1$	0	0	$-2e_1$	u_2
u_2	$-\frac{1}{2}u_2$	0	0	u_4	0
u_3	u_3	$2e_1$	$-u_4$	0	0
u_4	$\frac{1}{2}u_4$	$-u_2$	0	0	0

9.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	$\frac{1}{2}u_2$	$-u_3$	$-\frac{1}{2}u_4$
u_1	$-u_1$	0	0	0	u_2
u_2	$-\frac{1}{2}u_2$	0	0	0	0
u_3	u_3	0	0	0	0
u_4	$\frac{1}{2}u_4$	$-u_2$	0	0	0

$\lambda \in [0, 1]$

10.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	λu_2	$-u_3$	$-\lambda u_4$
u_1	$-u_1$	0	0	0	0
u_2	$-\lambda u_2$	0	0	0	0
u_3	u_3	0	0	0	0
u_4	λu_4	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 1.1^2 *is equivalent to one and only one of the following pairs:*

$\lambda = 0$

1.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	$-u_2$	u_1
u_2	0	0	0	0	$2u_2$
u_3	u_1	u_2	0	0	u_3
u_4	0	$-u_1$	$-2u_2$	$-u_3$	0

2.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	0	u_1
u_2	0	0	0	0	pu_2
u_3	u_1	0	0	0	u_3
u_4	0	$-u_1$	$-pu_2$	$-u_3$	0

3.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	e_1+u_2	0
u_2	0	0	0	0	0
u_3	u_1	$-e_1-u_2$	0	0	0
u_4	0	0	0	0	0

4.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	$-e_1+u_2$	0
u_2	0	0	0	0	0
u_3	u_1	e_1-u_2	0	0	0
u_4	0	0	0	0	0

5.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	u_2	0
u_2	0	0	0	0	0
u_3	u_1	$-u_2$	0	0	0
u_4	0	0	0	0	0

6.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	e_1	0
u_2	0	0	0	0	u_2
u_3	u_1	$-e_1$	0	0	0
u_4	0	0	$-u_2$	0	0

7.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	$-e_1$	0
u_2	0	0	0	0	u_2
u_3	u_1	e_1	0	0	0
u_4	0	0	$-u_2$	0	0

8.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	0	0
u_2	0	0	0	0	u_2
u_3	u_1	0	0	0	0
u_4	0	0	$-u_2$	0	0

9.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	e_1	0
u_2	0	0	0	0	0
u_3	u_1	$-e_1$	0	0	0
u_4	0	0	0	0	0

10.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	$-e_1$	0
u_2	0	0	0	0	0
u_3	u_1	e_1	0	0	0
u_4	0	0	0	0	0

$\lambda = \frac{1}{2}$

11.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	$\frac{1}{2}u_4$	$-u_1$	$-\frac{1}{2}u_2$
u_1	$-u_3$	0	u_2	$-4e_1$	$-u_4$
u_2	$-\frac{1}{2}u_4$	$-u_2$	0	$-u_4$	0
u_3	u_1	$4e_1$	u_4	0	u_2
u_4	$\frac{1}{2}u_2$	u_4	0	$-u_2$	0

$\lambda \in [0, 1]$

12.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	λu_4	$-u_1$	$-\lambda u_2$
u_1	$-u_3$	0	0	0	0
u_2	$-\lambda u_4$	0	0	0	0
u_3	u_1	0	0	0	0
u_4	λu_2	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 1.1^3 *is equivalent to one and only one pair:*

1.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	λu_4	$-u_3$	$-\lambda u_2$
u_1	$-u_1$	0	0	0	0
u_2	$-\lambda u_4$	0	0	0	0
u_3	u_3	0	0	0	0
u_4	λu_2	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 1.1^4 *is equivalent to one and only one pair:*

1.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	$-\lambda u_2$	$-u_1$	λu_4
u_1	$-u_3$	0	0	0	0
u_2	λu_2	0	0	0	0
u_3	u_1	0	0	0	0
u_4	$-\lambda u_4$	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 1.1^5 *is equivalent to one and only one pair:*

1.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	$\cos\frac{\phi}{2}u_1-\sin\frac{\phi}{2}u_2$	$\sin\frac{\phi}{2}u_1+\cos\frac{\phi}{2}u_2$	$-\cos\frac{\phi}{2}u_3+\sin\frac{\phi}{2}u_4$	$-\sin\frac{\phi}{2}u_3-\cos\frac{\phi}{2}u_4$
u_1	$-\cos\frac{\phi}{2}u_1+\sin\frac{\phi}{2}u_2$	0	0	0	0
u_2	$-\sin\frac{\phi}{2}u_1-\cos\frac{\phi}{2}u_2$	0	0	0	0
u_3	$\cos\frac{\phi}{2}u_3-\sin\frac{\phi}{2}u_4$	0	0	0	0
u_4	$\sin\frac{\phi}{2}u_3+\cos\frac{\phi}{2}u_4$	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 1.1^6 *is equivalent to one and only one pair:*

1.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	$-\sin\frac{\phi}{2}u_1-\cos\frac{\phi}{2}u_2$	$\cos\frac{\phi}{2}u_1-\sin\frac{\phi}{2}u_2$	$\sin\frac{\phi}{2}u_3-\cos\frac{\phi}{2}u_4$	$\cos\frac{\phi}{2}u_3+\sin\frac{\phi}{2}u_4$
u_1	$\sin\frac{\phi}{2}u_1+\cos\frac{\phi}{2}u_2$	0	0	0	0
u_2	$-\cos\frac{\phi}{2}u_1+\sin\frac{\phi}{2}u_2$	0	0	0	0
u_3	$-\sin\frac{\phi}{2}u_3+\cos\frac{\phi}{2}u_4$	0	0	0	0
u_4	$-\cos\frac{\phi}{2}u_3-\sin\frac{\phi}{2}u_4$	0	0	0	0

The proof of the Theorem follows from Propositions 1.1.1–1.1.10.

Proposition 1.1.1. *Any real form of the pair* 1.1.1 *is equivalent to one and only one of the following pairs:*

$$1.1^1.1, \quad 1.1^2.1.$$

Proof. 1. The group of automorphisms of the pair 1.1.1 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & ab & 0 & c \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & -ab & 0 & c \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b \in \mathbb{C}^*,\, c \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 1.1.1 has the form:

$$\mathcal{I}=\left\{ I=\begin{pmatrix}1&0&0&0&0\\0&\alpha&0&0&0\\0&0&\alpha\beta&0&\gamma\\0&0&0&\beta&0\\0&0&0&0&1\end{pmatrix}, \text{ or } \begin{pmatrix}-1&0&0&0&-1\\0&0&0&\beta&0\\0&0&-\alpha\beta&0&\gamma\\0&\alpha&0&0&0\\0&0&0&0&1\end{pmatrix} \;\middle|\; \begin{matrix}I\overline{I}=E,\\ \alpha,\beta\in\mathbb{C}^*,\\ \gamma\in\mathbb{C}\end{matrix} \right\}.$$

3. Let $A\in\mathcal{A}$, $I\in\mathcal{I}$. Using the operation $I\mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 1.1.1 is conjugate to one and only one of the following:

$$I_1=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\qquad I_2=\begin{pmatrix}-1&0&0&0&-1\\0&0&0&1&0\\0&0&-1&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X=(a,x,y,z,t)$, $I_2(X)=(-\overline{a}-\overline{t},\overline{z},-\overline{y},\overline{x},\overline{t})$, $I_2(X)=X$. It follows that $X=(a,x,y,\overline{x},-(a+\overline{a}))$, where $a,x\in\mathbb{C}$, $y\in\mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1'=ie_1,\ u_1'=u_1+u_3,\ u_2'=2iu_2,\ u_3'=iu_1-iu_3,\ u_4'=-e_1+2u_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $1.1^2.1$.

Proposition 1.1.2. *Any real form of the pair* 1.1.2 *is equivalent to one and only one of the following pairs:*

$$1.1^1.2,\quad 1.1^2.2.$$

Proof. 1. The group of automorphisms of the pair 1.1.2 has the form:

$$\mathcal{A}=\left\{ \begin{pmatrix}1&0&0&0&0\\0&a&0&0&0\\0&0&b&0&d\\0&0&0&c&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}-1&0&0&0&-1\\0&0&0&c&0\\0&0&b&0&d\\0&a&0&0&0\\0&0&0&0&1\end{pmatrix} \;\middle|\; a,b,c\in\mathbb{C}^*,\ d\in\mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 1.1.2 has the form:
if $p\in\mathbb{R}$

$$\mathcal{I}=\left\{ I=\begin{pmatrix}1&0&0&0&0\\0&\alpha&0&0&0\\0&0&\beta&0&\delta\\0&0&0&\gamma&0\\0&0&0&0&1\end{pmatrix}, \text{ or } \begin{pmatrix}-1&0&0&0&-1\\0&0&0&\gamma&0\\0&0&\beta&0&\delta\\0&\alpha&0&0&0\\0&0&0&0&1\end{pmatrix} \;\middle|\; \begin{matrix}I\overline{I}=E,\\ \alpha,\beta,\gamma\in\mathbb{C}^*,\\ \delta\in\mathbb{C}\end{matrix} \right\}.$$

For other values of p there exist no anti–involution.

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 1.1.2 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, x, y, z, t)$, $I_2(X) = (-\overline{a} - \overline{t}, \overline{z}, -\overline{y}, \overline{x}, \overline{t})$, $I_2(X) = X$. It follows that $X = (a, x, y, \overline{x}, -(a + \overline{a}))$, where $a, x \in \mathbb{C}$, $y \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1' = ie_1,\ u_1' = u_1 + u_3,\ u_2' = 2iu_2,\ u_3' = iu_1 - iu_3,\ u_4' = -e_1 + 2u_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $1.1^2.2$.

Proposition 1.1.3. *Any real form of the pair* 1.1.3 *is equivalent to one and only one of the following pairs:*

$$1.1^1.3, \quad 1.1^2.3, \quad 1.1^2.4.$$

Proof. 1. The group of automorphisms of the pair 1.1.3 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & -1 & 0 & c \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix} \middle|\ a, b \in \mathbb{C}^*,\ c \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 1.1.3 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \gamma \\ 0 & 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & 0 & \beta \end{pmatrix}, \text{ or } \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & -1 & 0 & \gamma \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta \end{pmatrix} \middle|\ \begin{array}{c} I\overline{I} = E, \\ \alpha, \beta \in \mathbb{C}^*, \\ \gamma \in \mathbb{C} \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 1.1.3 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$I_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, x, y, z, t)$, $I_2(X) = (-\overline{a}, -\overline{z}, -\overline{y}, -\overline{x}, \overline{t})$, $I_2(X) = X$. It follows that $X = (a, x, y, -\overline{x}, t)$, where $x \in \mathbb{C}$, $t \in \mathbb{R}$, $a, y \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1' = ie_1,\ u_1' = \frac{1}{\sqrt{2}}u_1 - \frac{1}{\sqrt{2}}u_3,\ u_2' = iu_2,\ u_3' = \frac{i}{\sqrt{2}}u_1 + \frac{i}{\sqrt{2}}u_3,\ u_4' = u_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $1.1^2.3$.

5. Find the set of fixed points of the mapping I_3.

Let $X = (a, x, y, z, t)$, $I_3(X) = (-\overline{a}, \overline{z}, -\overline{y}, \overline{x}, \overline{t})$, $I_3(X) = X$. It follows that $X = (a, x, y, \overline{x}, t)$, where $x \in \mathbb{C}$, $t \in \mathbb{R}$, $a, y \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1' = ie_1,\ u_1' = \frac{1}{\sqrt{2}}u_1 + \frac{1}{\sqrt{2}}u_3,\ u_2' = -iu_2,\ u_3' = \frac{i}{\sqrt{2}}u_1 - \frac{i}{\sqrt{2}}u_3,\ u_4' = u_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_3 has the form $1.1^2.4$.

Proposition 1.1.4. *Any real form of the pair* 1.1.4 *is equivalent to one and only one of the following pairs:*

$$1.1^1.4, \quad 1.1^2.5.$$

Proof. 1. The group of automorphisms of the pair 1.1.4 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & ab & 0 & d \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & c \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & -ab & 0 & d \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c \end{pmatrix} \middle|\ a, b, c \in \mathbb{C}^*,\ d \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 1.1.4 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha\beta & 0 & \delta \\ 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & \gamma \end{pmatrix}, \text{or} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & -\alpha\beta & 0 & \delta \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma \end{pmatrix} \middle|\ \begin{matrix} I\overline{I} = E, \\ \alpha, \beta, \gamma \in \mathbb{C}^*, \\ \delta \in \mathbb{C} \end{matrix} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 1.1.4 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, x, y, z, t)$, $I_2(X) = (-\overline{a}, \overline{z}, -\overline{y}, \overline{x}, \overline{t})$, $I_2(X) = X$. It follows that $X = (a, x, y, \overline{x}, t)$, where $x \in \mathbb{C}$, $t \in \mathbb{R}$, $a, y \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1' = ie_1,\ u_1' = \frac{1}{\sqrt{2}}u_1 + \frac{1}{\sqrt{2}}u_3,\ u_2' = -iu_2,\ u_3' = \frac{i}{\sqrt{2}}u_1 - \frac{i}{\sqrt{2}}u_3,\ u_4' = u_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $1.1^2.5$.

Proposition 1.1.5. *Any real form of the pair* 1.1.5 *is equivalent to one and only one of the following pairs:*

$$1.1^1.5, \quad 1.1^2.6, \quad 1.1^2.7.$$

Proof. 1. The group of automorphisms of the pair 1.1.5 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & c \\ 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & b & 0 & c \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \ a, b \in \mathbb{C}^*,\ c \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 1.1.5 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & \gamma \\ 0 & 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{ or } \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & \beta & 0 & \gamma \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \begin{array}{c} I\overline{I} = E, \\ \alpha, \beta \in \mathbb{C}^*, \\ \gamma \in \mathbb{C} \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 1.1.5 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$I_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, x, y, z, t)$, $I_2(X) = (-\overline{a}, -\overline{z}, -\overline{y}, -\overline{x}, \overline{t})$, $I_2(X) = X$. It follows that $X = (a, x, y, -\overline{x}, t)$, where $x \in \mathbb{C}$, $t \in \mathbb{R}$, $a, y \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1' = ie_1,\ u_1' = \frac{1}{\sqrt{2}}u_1 - \frac{1}{\sqrt{2}}u_3,\ u_2' = -iu_2,\ u_3' = \frac{i}{\sqrt{2}}u_1 + \frac{i}{\sqrt{2}}u_3,\ u_4' = u_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $1.1^2.6$.

5. Find the set of fixed points of the mapping I_3.

Let $X = (a, x, y, z, t)$, $I_3(X) = (-\overline{a}, \overline{z}, -\overline{y}, \overline{x}, \overline{t})$, $I_3(X) = X$. It follows that $X = (a, x, y, \overline{x}, t)$, where $x \in \mathbb{C}$, $t \in \mathbb{R}$, $a, y \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1' = ie_1,\ u_1' = \frac{1}{\sqrt{2}}u_1 + \frac{1}{\sqrt{2}}u_3,\ u_2' = -iu_2,\ u_3' = \frac{i}{\sqrt{2}}u_1 - \frac{i}{\sqrt{2}}u_3,\ u_4' = u_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_3 has the form $1.1^2.7$.

Proposition 1.1.6. *Any real form of the pair* 1.1.6 *is equivalent to one and only one of the following pairs:*

$$1.1^1.6, \quad 1.1^2.8.$$

Proof. 1. The group of automorphisms of the pair 1.1.6 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & d \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & b & 0 & d \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \ a, b, c \in \mathbb{C}^*,\ d \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 1.1.6 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & \delta \\ 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{ or } \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & \beta & 0 & \delta \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \ \begin{matrix} I\overline{I} = E, \\ \alpha, \beta, \gamma \in \mathbb{C}^*, \\ \delta \in \mathbb{C} \end{matrix} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 1.1.6 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, x, y, z, t)$, $I_2(X) = (-\overline{a}, \overline{z}, -\overline{y}, \overline{x}, \overline{t})$, $I_2(X) = X$. It follows that $X = (a, x, y, \overline{x}, t)$, where $x \in \mathbb{C}$, $t \in \mathbb{R}$, $a, y \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1' = ie_1,\ u_1' = \frac{1}{\sqrt{2}}u_1 + \frac{1}{\sqrt{2}}u_3,\ u_2' = -iu_2,\ u_3' = \frac{i}{\sqrt{2}}u_1 - \frac{i}{\sqrt{2}}u_3,\ u_4' = u_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $1.1^2.8$.

Proposition 1.1.7. *Any real form of the pair 1.1.7 is equivalent to one and only one of the following pairs:*

$$1.1^1.7, \quad 1.1^2.9, \quad 1.1^2.10.$$

Proof. 1. The group of automorphisms of the pair 1.1.7 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b_1 & 0 & c_1 \\ 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & b_2 & 0 & c_2 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & b_1 & 0 & c_1 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b_2 & 0 & c_2 \end{pmatrix} \;\middle|\; \begin{array}{c} a \in \mathbb{C}^*, \\ b_1, b_2, c_1, c_2 \in \mathbb{C}, \\ b_1c_2 - c_1b_2 \neq 0 \end{array} \right\}.$$

2. The set of anti–involutions of the pair 1.1.7 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & \gamma_1 \\ 0 & 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & \beta_2 & 0 & \gamma_2 \end{pmatrix}, \text{or} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & \beta_1 & 0 & \gamma_1 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta_2 & 0 & \gamma_2 \end{pmatrix} \;\middle|\; \begin{array}{c} I\overline{I} = E,\ \alpha \in \mathbb{C}^*, \\ \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{C}, \\ \beta_1\gamma_2 - \beta_2\gamma_1 \neq 0 \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 1.1.7 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$I_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, x, y, z, t)$, $I_2(X) = (-\overline{a}, -\overline{z}, -\overline{y}, -\overline{x}, \overline{t})$, $I_2(X) = X$. It follows that $X = (a, x, y, -\overline{x}, t)$, where $x \in \mathbb{C}$, $t \in \mathbb{R}$, $a, y \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1' = ie_1,\ u_1' = \frac{1}{\sqrt{2}}u_1 - \frac{1}{\sqrt{2}}u_3,\ u_2' = -iu_2,\ u_3' = \frac{i}{\sqrt{2}}u_1 + \frac{i}{\sqrt{2}}u_3,\ u_4' = u_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $1.1^2.9$.

5. Find the set of fixed points of the mapping I_3.

Let $X = (a, x, y, z, t)$, $I_3(X) = (-\overline{a}, \overline{z}, -\overline{y}, \overline{x}, \overline{t})$, $I_3(X) = X$. It follows that $X = (a, x, y, \overline{x}, t)$, where $x \in \mathbb{C}$, $t \in \mathbb{R}$, $a, y \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1' = ie_1,\ u_1' = \frac{1}{\sqrt{2}}u_1 + \frac{1}{\sqrt{2}}u_3,\ u_2' = -iu_2,\ u_3' = \frac{i}{\sqrt{2}}u_1 - \frac{i}{\sqrt{2}}u_3,\ u_4' = u_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_3 has the form $1.1^2.10$.

Proposition 1.1.8. *Any real form of the pair* 1.1.8 *is equivalent to one and only one of the following pairs:*

$$1.1^1.8, \quad 1.1^2.11.$$

Proof. 1. The group of automorphisms of the pair 1.1.8 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & ab & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & 0 & b \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & -ab & 0 & 0 \end{pmatrix} \middle| \; a, b \in \mathbb{C}^* \right\}.$$

2. The set of anti–involutions of the pair 1.1.8 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha\beta & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & 0 & \beta \end{pmatrix}, \text{ or } \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & 0 & \beta \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & -\alpha\beta & 0 & 0 \end{pmatrix} \middle| \; \begin{matrix} I\overline{I} = E, \\ \alpha, \beta \in \mathbb{C}^* \end{matrix} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 1.1.8 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, x, y, z, t)$, $I_2(X) = (-\overline{a}, -\overline{z}, \overline{t}, -\overline{x}, \overline{y})$, $I_2(X) = X$. It follows that $X = (a, x, y, -\overline{x}, \overline{y})$, where $x, y \in \mathbb{C}$, $a \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1' = ie_1, \; u_1' = u_1 - u_3, \; u_2' = u_2 + u_4, \; u_3' = iu_1 + iu_3, \; u_4' = iu_2 - iu_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $1.1^2.11$.

Proposition 1.1.9. *Any real form of the pair* 1.1.9 *is equivalent to one and only one pair:*

$$1.1^1.9.$$

Proof. 1. The group of automorphisms of the pair 1.1.9 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & ab & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix} \middle| \; a, b, c \in \mathbb{C}^* \right\}.$$

2. The set of anti–involutions of the pair 1.1.9 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha\beta & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & \beta \end{pmatrix} \;\middle|\; \begin{array}{c} I\bar{I} = E, \\ \alpha, \beta, \gamma \in \mathbb{C}^* \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\bar{A}^{-1}$ we obtain that any anti–involution of the pair 1.1.9 is conjugate to one and only one:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 1.1.10. *Any real form of the pair* 1.1.10 *is equivalent to one and only one of the following pairs:*

$$\begin{array}{ll} 1.1^1.10\ (\operatorname{Re}\lambda \in [0,1], \operatorname{Im}\lambda = 0), & 1.1^2.12\ (\operatorname{Re}\lambda \in [0,1], \operatorname{Im}\lambda = 0), \\ 1.1^3.1\ (\operatorname{Im}\lambda \in]0,1], \operatorname{Re}\lambda = 0), & 1.1^4.1\ (\operatorname{Im}\lambda \in]0,1[, \operatorname{Re}\lambda = 0), \\ 1.1^5.1\ (|\lambda| = 1, \arg\lambda \in]0, \frac{\pi}{2}]\,), & 1.1^6.1\ (|\lambda| = 1, \arg\lambda \in]0, \frac{\pi}{2}[\,). \end{array}$$

Proof. Consider the following cases:

1°. $\lambda = 0$.

1. The group of automorphisms of the pair 1.1.10 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b_1 & 0 & c_1 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & b_2 & 0 & c_2 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & b_1 & 0 & c_1 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b_2 & 0 & c_2 \end{pmatrix} \;\middle|\; \begin{array}{c} a, d \in \mathbb{C}^*, \\ b_1, b_2, c_1, c_2 \in \mathbb{C}, \\ b_1c_2 - c_1b_2 \neq 0 \end{array} \right\}.$$

2. The set of anti–involutions of the pair 1.1.10 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & \gamma_1 \\ 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & \beta_2 & 0 & \gamma_2 \end{pmatrix}, \text{or} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & \beta_1 & 0 & \gamma_1 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta_2 & 0 & \gamma_2 \end{pmatrix} \;\middle|\; \begin{array}{c} I\bar{I} = E,\ \alpha, \delta \in \mathbb{C}^*, \\ \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{C}, \\ \beta_1\gamma_2 - \beta_2\gamma_1 \neq 0 \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\bar{A}^{-1}$ we obtain that any anti–involution of the pair 1.1.10 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, x, y, z, t)$, $I_2(X) = (-\overline{a}, \overline{z}, -\overline{y}, \overline{x}, \overline{t})$, $I_2(X) = X$. It follows that $X = (a, x, y, \overline{x}, t)$, where $x \in \mathbb{C}$, $t \in \mathbb{R}$, $a, y \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1' = ie_1,\ u_1' = \frac{1}{\sqrt{2}}u_1 + \frac{1}{\sqrt{2}}u_3,\ u_2' = -iu_2,\ u_3' = \frac{i}{\sqrt{2}}u_1 - \frac{i}{\sqrt{2}}u_3,\ u_4' = u_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $1.1^2.12$ $(\lambda = 0)$.

2°. $\lambda = 1$.

1. The group of automorphisms of the pair 1.1.10 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a_1 & b_1 & 0 & 0 \\ 0 & a_2 & b_2 & 0 & 0 \\ 0 & 0 & 0 & c_1 & d_1 \\ 0 & 0 & 0 & c_2 & d_2 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & d_1 \\ 0 & 0 & 0 & c_2 & d_2 \\ 0 & a_1 & b_1 & 0 & 0 \\ 0 & a_2 & b_2 & 0 & 0 \end{pmatrix} \middle| \begin{array}{c} a_1, a_2, b_1, b_2, \\ c_1, c_2, d_1, d_2 \in \mathbb{C}, \\ a_1 b_2 - b_1 a_2 \neq 0, \\ c_1 d_2 - d_1 c_2 \neq 0 \end{array} \right\}.$$

2. The set of anti–involutions of the pair 1.1.10 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & \alpha_2 & \beta_2 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & \delta_1 \\ 0 & 0 & 0 & \gamma_2 & \delta_2 \end{pmatrix}, \text{ or } \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & \delta_1 \\ 0 & 0 & 0 & \gamma_2 & \delta_2 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & \alpha_2 & \beta_2 & 0 & 0 \end{pmatrix} \middle| \begin{array}{c} I\overline{I} = E,\ \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{C}, \\ \alpha_1\beta_2 - \beta_1\alpha_2 \neq 0, \gamma_1\delta_2 - \delta_1\gamma_2 \neq 0 \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 1.1.10 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, x, y, z, t)$, $I_2(X) = (-\overline{a}, \overline{z}, \overline{t}, \overline{x}, \overline{y})$, $I_2(X) = X$. It follows that $X = (a, x, y, \overline{x}, \overline{y})$, where $x, y \in \mathbb{C}$, $a \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1' = ie_1,\ u_1' = u_1 + u_3,\ u_2' = u_2 + u_4,\ u_3' = iu_1 - iu_3,\ u_4' = iu_2 - iu_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $1.1^2.12$ $(\lambda = 1)$.

3°. $\lambda = i$.

1. The group of automorphisms of the pair 1.1.10 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & d \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & d \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} \frac{1}{i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & c & 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{i} & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & d \\ 0 & a & 0 & 0 & 0 \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{C}^* \right\}.$$

2. The set of anti–involutions of the pair 1.1.10 has the form:

$$\mathcal{I} = \left\{ I \in \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & \beta & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & \beta & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta \end{pmatrix}, \right. \right.$$
$$\left. \left. \begin{pmatrix} \frac{1}{i} & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & \gamma & 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & \beta & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \end{pmatrix} \right\} \middle| \; \begin{matrix} I\overline{I} = E, \\ \alpha, \beta, \gamma, \delta \in \mathbb{C}^* \end{matrix} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 1.1.10 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} \frac{1}{i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_1.

Let $X = (a, x, y, z, t)$, $I_1(X) = (\overline{a}, \overline{x}, \overline{t}, \overline{z}, \overline{y})$, $I_1(X) = X$. It follows that $X = (a, x, y, z, \overline{y})$, where $y \in \mathbb{C}$, $a, x, z \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$e_1' = e_1, \; u_1' = u_1, \; u_2' = u_2 + u_4, \; u_3' = u_3, \; u_4' = iu_2 - iu_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_1 has the form $1.1^3.1$ $(\lambda = 1)$.

5. Find the set of fixed points of the mapping I_2.

Let $X = (a, x, y, z, t)$, $I_2(X) = (\frac{\overline{a}}{i}, \overline{y}, \overline{x}, \overline{t}, \overline{z})$, $I_2(X) = X$. It follows that $X = ((\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})a, x, \overline{x}, z, \overline{z})$, where $x, z \in \mathbb{C}$, $a \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$e_1' = (\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})e_1,\ u_1' = u_1 + u_2,\ u_2' = iu_1 - iu_2,\ u_3' = u_3 + u_4,\ u_4' = iu_3 - iu_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $1.1^5.1$ $(\phi = \frac{\pi}{2})$.

$4°$. $\operatorname{Re}\lambda \in]0,1[$, $\operatorname{Im}\lambda = 0$.

1. The group of automorphisms of the pair 1.1.10 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & d \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & d \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \end{pmatrix} \middle|\ a, b, c, d \in \mathbb{C}^* \right\}.$$

2. The set of anti–involutions of the pair 1.1.10 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & \delta \end{pmatrix}, \text{ or } \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & \delta \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 \end{pmatrix} \middle|\ \begin{matrix} I\overline{I} = E, \\ \alpha, \beta, \gamma, \delta \in \mathbb{C}^* \end{matrix} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 1.1.10 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, x, y, z, t)$, $I_2(X) = (-\overline{a}, \overline{z}, \overline{t}, \overline{x}, \overline{y})$, $I_2(X) = X$. It follows that $X = (a, x, y, \overline{x}, \overline{y})$, where $x, y \in \mathbb{C}$, $a \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1' = ie_1,\ u_1' = u_1 + u_3,\ u_2' = u_2 + u_4,\ u_3' = iu_1 - iu_3,\ u_4' = iu_2 - iu_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $1.1^2.12$ $(\lambda \in]0,1[)$.

$5°$. $\operatorname{Im}\lambda \in]0,1[$, $\operatorname{Re}\lambda = 0$.

1. The group of automorphisms of the pair 1.1.10 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & d \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & d \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \end{pmatrix} \middle|\ a, b, c, d \in \mathbb{C}^* \right\}.$$

2. The set of anti–involutions of the pair 1.1.10 has the form:

$$\mathcal{I}=\left\{ I=\begin{pmatrix}1&0&0&0&0\\0&\alpha&0&0&0\\0&0&0&0&\delta\\0&0&0&\gamma&0\\0&0&\beta&0&0\end{pmatrix},\text{ or }\begin{pmatrix}-1&0&0&0&0\\0&0&0&\gamma&0\\0&0&\beta&0&0\\0&\alpha&0&0&0\\0&0&0&0&\delta\end{pmatrix}\;\middle|\; \begin{array}{c}I\overline{I}=E,\\ \alpha,\beta,\gamma,\delta\in\mathbb{C}^*\end{array}\right\}.$$

3. Let $A\in\mathcal{A}$, $I\in\mathcal{I}$. Using the operation $I\mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 1.1.10 is conjugate to one and only one of the following:

$$I_1=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix},\qquad I_2=\begin{pmatrix}-1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_1.

Let $X=(a,x,y,z,t)$, $I_1(X)=(\overline{a},\overline{x},\overline{t},\overline{z},\overline{y})$, $I_1(X)=X$. It follows that $X=(a,x,y,z,\overline{y})$, where $y\in\mathbb{C}$, $a,x,z\in\mathbb{R}$.

Consider the following basis of the set of fixed points:

$$e_1'=e_1,\ u_1'=u_1,\ u_2'=u_2+u_4,\ u_3'=u_3,\ u_4'=iu_2-iu_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_1 has the form $1.1^3.1$ ($\lambda\in]0,1[$).

5. Find the set of fixed points of the mapping I_2.

Let $X=(a,x,y,z,t)$, $I_2(X)=(-\overline{a},\overline{z},\overline{y},\overline{x},\overline{t})$, $I_2(X)=X$. It follows that $X=(a,x,y,\overline{x},t)$, where $x\in\mathbb{C}$, $a\in\mathbb{R}i$, $y,t\in\mathbb{R}$.

Consider the following basis of the set of fixed points:

$$e_1'=ie_1,\ u_1'=u_1+u_3,\ u_2'=u_2,\ u_3'=iu_1-iu_3,\ u_4'=u_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $1.1^4.1$.

6°. $|\lambda|=1$, $\arg\lambda\in]0,\frac{\pi}{2}[$.

1. The group of automorphisms of the pair 1.1.10 has the form:

$$\mathcal{A}=\left\{\begin{pmatrix}1&0&0&0&0\\0&a&0&0&0\\0&0&b&0&0\\0&0&0&c&0\\0&0&0&0&d\end{pmatrix},\begin{pmatrix}-1&0&0&0&0\\0&0&0&c&0\\0&0&0&0&d\\0&a&0&0&0\\0&0&b&0&0\end{pmatrix}\;\middle|\; a,b,c,d\in\mathbb{C}^*\right\}.$$

2. The set of anti–involutions of the pair 1.1.10 has the form:

$$\mathcal{I}=\left\{ I=\begin{pmatrix}\frac{1}{\lambda}&0&0&0&0\\0&0&\beta&0&0\\0&\alpha&0&0&0\\0&0&0&0&\delta\\0&0&0&\gamma&0\end{pmatrix},\text{ or }\begin{pmatrix}-\frac{1}{\lambda}&0&0&0&0\\0&0&0&0&\delta\\0&0&0&\gamma&0\\0&0&\beta&0&0\\0&\alpha&0&0&0\end{pmatrix}\;\middle|\; \begin{array}{c}I\overline{I}=E,\\ \alpha,\beta,\gamma,\delta\in\mathbb{C}^*\end{array}\right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 1.1.10 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} -\frac{1}{\lambda} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_1.

Let $X = (a, x, y, z, t)$, $I_1(X) = (\frac{\overline{a}}{\lambda}, \overline{y}, \overline{x}, \overline{t}, \overline{z})$, $I_1(X) = X$. It follows that $X = (ae^{-\frac{i\phi}{2}}, x, \overline{x}, z, \overline{z})$, where $x, z \in \mathbb{C}$, $a \in \mathbb{R}$, $\phi = \arg \lambda$.

Consider the following basis of the set of fixed points:

$$e_1' = e^{-\frac{i\phi}{2}} e_1,\ u_1' = u_1 + u_2,\ u_2' = iu_1 - iu_2,\ u_3' = u_3 + u_4,\ u_4' = iu_3 - iu_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_1 has the form $1.1^5.1$ ($\phi \in]0, \frac{\pi}{2}[$).

5. Find the set of fixed points of the mapping I_2.

Let $X = (a, x, y, z, t)$, $I_2(X) = (-\frac{\overline{a}}{\lambda}, \overline{t}, \overline{z}, \overline{y}, \overline{x})$, $I_2(X) = X$. It follows that $X = (ae^{-\frac{i(\phi+\pi)}{2}}, x, y, \overline{y}, \overline{x})$, where $x, y \in \mathbb{C}$, $a \in \mathbb{R}$, $\phi = \arg \lambda$.

Consider the following basis of the set of fixed points:

$$e_1' = e^{-\frac{i(\phi+\pi)}{2}} e_1,\ u_1' = u_1 + u_4,\ u_2' = iu_1 - iu_4,\ u_3' = u_2 + u_3,\ u_4' = iu_2 - iu_3.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $1.1^6.1$.

For other values of λ there exist no anti–involution.

Theorem 1.2. *Any real form of the linear Lie algebra* 1.2 *is conjugate to one and only one of the following linear Lie algebras:*

$$1.2^1 \quad \begin{pmatrix} x & x & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -x & -x \end{pmatrix} \qquad 1.2^2 \quad \begin{pmatrix} 0 & -x & 0 & -x \\ x & 0 & x & 0 \\ 0 & 0 & 0 & -x \\ 0 & 0 & x & 0 \end{pmatrix}$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 1.2^1 *is equivalent to one and only one pair:*

1.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	$u_1{+}u_2$	$-u_3{-}u_4$	$-u_4$
u_1	$-u_1$	0	0	0	0
u_2	$-u_1{-}u_2$	0	0	0	0
u_3	$u_3{+}u_4$	0	0	0	0
u_4	u_4	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 1.2^2 *is equivalent to one and only one pair:*

1.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_2	$-u_1$	u_2+u_4	$-u_1-u_3$
u_1	$-u_2$	0	0	0	0
u_2	u_1	0	0	0	0
u_3	$-u_2-u_4$	0	0	0	0
u_4	u_1+u_3	0	0	0	0

Proof. 1. The group of automophisms of the pair 1.2.1 has the form:

$$\mathcal{A}=\left\{\begin{pmatrix}1&0&0&0&0\\0&a&c&0&0\\0&0&a&0&0\\0&0&0&b&0\\0&0&0&d&b\end{pmatrix},\begin{pmatrix}-1&0&0&0&0\\0&0&0&d&b\\0&0&0&b&0\\0&0&a&0&0\\0&a&c&0&0\end{pmatrix}\;\middle|\; a,b\in\mathbb{C}^*,\ c,d\in\mathbb{C}\right\}.$$

2. The set of anti–involutions of the pair 1.2.1 has the form:

$$\mathcal{I}=\left\{I=\begin{pmatrix}1&0&0&0&0\\0&\alpha&\gamma&0&0\\0&0&\alpha&0&0\\0&0&0&\beta&0\\0&0&0&\delta&\beta\end{pmatrix},\text{ or }\begin{pmatrix}-1&0&0&0&0\\0&0&0&\delta&\beta\\0&0&0&\beta&0\\0&0&\alpha&0&0\\0&\alpha&\gamma&0&0\end{pmatrix}\;\middle|\;\begin{matrix}I\bar{I}=E,\\ \alpha,\beta\in\mathbb{C}^*,\\ \gamma,\delta\in\mathbb{C}\end{matrix}\right\}.$$

3. Let $A\in\mathcal{A}$, $I\in\mathcal{I}$. Using the operation $I\mapsto AI\bar{A}^{-1}$ we obtain that any anti–involution of the pair 1.2.1 is conjugate to one and only one of the following:

$$I_1=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\qquad I_2=\begin{pmatrix}-1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X=(a,x,y,z,t)$, $I_2(X)=(-\bar{a},\bar{t},\bar{z},\bar{y},\bar{x})$, $I_2(X)=X$. It follows that $X=(a,x,y,\bar{y},\bar{x})$, where $x,y\in\mathbb{C}$, $a\in\mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1'=ie_1,\ u_1'=u_1+u_4,\ u_2'=iu_2-iu_4,\ u_3'=u_2+u_3,\ u_4'=iu_2-iu_3.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $1.2^2.1$.

Theorem 1.3. *Any real form of the linear Lie algebra 1.3 is conjugate to one and only one linear Lie algebra:*

$$1.3^1 \quad \begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 1.3^1 *is equivalent to one and only one of the following pairs:*

1.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	e_1	0	u_1	u_2
u_1	$-e_1$	0	$-\frac{1}{2}u_2$	u_3	$\frac{1}{2}u_4$
u_2	0	$\frac{1}{2}u_2$	0	$\frac{1}{2}u_4$	0
u_3	$-u_1$	$-u_3$	$-\frac{1}{2}u_4$	0	0
u_4	$-u_2$	$-\frac{1}{2}u_4$	0	0	0

2.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-\lambda e_1+(\lambda+1)u_1+\lambda u_2$	0
u_2	0	0	0	0	u_2
u_3	$-u_1$	$\lambda e_1-(\lambda+1)u_1-\lambda u_2$	0	0	0
u_4	$-u_2$	0	$-u_2$	0	0

$$|\lambda| \leqslant 1$$

3.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	u_1	0
u_2	0	0	0	0	u_2
u_3	$-u_1$	$-u_1$	0	0	e_1
u_4	$-u_2$	0	$-u_2$	$-e_1$	0

4.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-(1+\lambda^2)e_1+2\lambda u_1+(1+\lambda^2)u_2$	0
u_2	0	0	0	0	u_2
u_3	$-u_1$	$(1+\lambda^2)e_1-2\lambda u_1-(1+\lambda^2)u_2$	0	0	0
u_4	$-u_2$	0	$-u_2$	0	0

$$\lambda \geqslant 0$$

5.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-\frac{\lambda^2+\mu}{\mu-1}e_1+\frac{1+\lambda^2}{\mu-1}u_2$	$-\lambda e_1+u_1+\lambda u_2$
u_2	0	0	0	$-\lambda e_1+u_1+\lambda u_2$	$-\mu e_1+(\mu+1)u_2$
u_3	$-u_1$	$\frac{\lambda^2+\mu}{\mu-1}e_1-\frac{1+\lambda^2}{\mu-1}u_2$	$\lambda e_1-u_1-\lambda u_2$	0	0
u_4	$-u_2$	$\lambda e_1-u_1-\lambda u_2$	$\mu e_1-(\mu+1)u_2$	0	0

$$|\lambda|\geqslant 0,\ \mu\neq 1$$

6.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-u_2$	u_1
u_2	0	0	0	u_1	u_2
u_3	$-u_1$	u_2	$-u_1$	0	e_1
u_4	$-u_2$	$-u_1$	$-u_2$	$-e_1$	0

7.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	x	y
u_2	0	0	0	y	z
u_3	$-u_1$	$-x$	$-y$	0	0
u_4	$-u_2$	$-y$	$-z$	0	0,

where

$$x=\frac{1}{1+\lambda}e_1+\frac{\lambda}{1+\lambda}u_1-\frac{1}{1+\lambda}u_2,$$
$$y=-\frac{1}{1+\lambda}e_1+\frac{1}{1+\lambda}u_1+\frac{1}{1+\lambda}u_2,$$
$$z=-\frac{\lambda}{1+\lambda}e_1+\frac{\lambda}{1+\lambda}u_1+\frac{1+2\lambda}{1+\lambda}u_2,$$
$$\lambda\neq -1$$

8.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	u_1	u_2
u_3	$-u_1$	0	$-u_1$	0	$-u_3$
u_4	$-u_2$	0	$-u_2$	u_3	0

9.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	λu_1	$-\lambda e_1+(\lambda+1)u_2$
u_3	$-u_1$	0	$-\lambda u_1$	0	$-\lambda u_3$
u_4	$-u_2$	0	$\lambda e_1-(\lambda+1)u_2$	λu_3	0

10.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	u_2
u_3	$-u_1$	0	0	0	e_1
u_4	$-u_2$	0	$-u_2$	$-e_1$	0

11.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	$-u_1$	e_1
u_3	$-u_1$	0	u_1	0	e_1+u_3
u_4	$-u_2$	0	$-e_1$	$-e_1-u_3$	0

12.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	μu_1	$-\lambda\mu e_1+(\lambda+\mu)u_2$
u_3	$-u_1$	0	$-\mu u_1$	0	$(1-\mu)u_3$
u_4	$-u_2$	$-u_1$	$\lambda\mu e_1-(\lambda+\mu)u_2$	$(\mu-1)u_3$	0

13.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	$\frac{1}{2}u_1$	$-\frac{\lambda}{2}e_1+(\lambda+\frac{1}{2})u_2$
u_3	$-u_1$	0	$-\frac{1}{2}u_1$	0	$e_1+\frac{1}{2}u_3$
u_4	$-u_2$	$-u_1$	$\frac{\lambda}{2}e_1-(\lambda+\frac{1}{2})u_2$	$-e_1-\frac{1}{2}u_3$	0

14.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	$(1-\lambda)u_1$	$\lambda(\lambda-1)e_1+u_2$
u_3	$-u_1$	0	$(\lambda-1)u_1$	0	$e_1+\lambda u_3$
u_4	$-u_2$	$-u_1$	$\lambda(1-\lambda)e_1-u_2$	$-e_1-\lambda u_3$	0

$\lambda \neq \frac{1}{2}$

15.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-e_1+2u_1$	u_2
u_2	0	0	0	u_2	$-e_1+u_1$
u_3	$-u_1$	e_1-2u_1	$-u_2$	0	0
u_4	$-u_2$	$-u_2$	e_1-u_1	0	0

16.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-e_1+2u_1$	u_2
u_2	0	0	0	u_2	e_1-u_1
u_3	$-u_1$	e_1-2u_1	$-u_2$	0	0
u_4	$-u_2$	$-u_2$	$-e_1+u_1$	0	0

17.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	u_1
u_3	$-u_1$	0	0	0	e_1
u_4	$-u_2$	0	$-u_1$	$-e_1$	0

18.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	u_1
u_3	$-u_1$	0	0	0	0
u_4	$-u_2$	0	$-u_1$	0	0

19.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	u_1	$-e_1+u_1+2u_2$
u_3	$-u_1$	0	$-u_1$	0	0
u_4	$-u_2$	$-u_1$	$e_1-u_1-2u_2$	0	0

20.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	u_1	u_2-u_1
u_3	$-u_1$	0	$-u_1$	0	$-u_3$
u_4	$-u_2$	0	u_1-u_2	u_3	0

21.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	λu_1	$-\lambda e_1+(1-\lambda)u_1+(1+\lambda)u_2$
u_3	$-u_1$	0	$-\lambda u_1$	0	$(1-\lambda)u_3$
u_4	$-u_2$	$-u_1$	$\lambda e_1+(\lambda-1)u_1-(1+\lambda)u_2$	$(\lambda-1)u_3$	0

$$\lambda \neq 1$$

22.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	$\frac{1}{2}u_1$	$-\frac{1}{2}e_1+\frac{1}{2}u_1+\frac{3}{2}u_2$
u_3	$-u_1$	0	$-\frac{1}{2}u_1$	0	$e_1+\frac{1}{2}u_3$
u_4	$-u_2$	$-u_1$	$\frac{1}{2}e_1-\frac{1}{2}u_1-\frac{3}{2}u_2$	$-e_1-\frac{1}{2}u_3$	0

23.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	0	u_1+u_2
u_3	$-u_1$	0	0	0	e_1+u_3
u_4	$-u_2$	$-u_1$	$-u_1-u_2$	$-e_1-u_3$	0

24.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$(1-2\lambda)e_1+2\lambda u_1$	$(2\lambda-1)u_2$
u_2	0	0	0	λu_2	$\frac{2\lambda-1}{2(\lambda-1)}e_1-\frac{1}{2(\lambda-1)}u_1$
u_3	$-u_1$	$(2\lambda-1)e_1-2\lambda u_1$	$-\lambda u_2$	0	$(\lambda-1)u_4$
u_4	$-u_2$	$(1-2\lambda)u_2$	$\frac{1-2\lambda}{2(\lambda-1)}e_1+\frac{1}{2(\lambda-1)}u_1$	$(1-\lambda)u_4$	0

$$\lambda \neq 1$$

25.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$(1-2\lambda)e_1+2\lambda u_1$	$(2\lambda-1)u_2$
u_2	0	0	0	λu_2	$\frac{1-2\lambda}{2(\lambda-1)}e_1+\frac{1}{2(\lambda-1)}u_1$
u_3	$-u_1$	$(2\lambda-1)e_1-2\lambda u_1$	$-\lambda u_2$	0	$(\lambda-1)u_4$
u_4	$-u_2$	$(1-2\lambda)u_2$	$-\frac{1-2\lambda}{2(\lambda-1)}e_1-\frac{1}{2(\lambda-1)}u_1$	$(1-\lambda)u_4$	0

$$\lambda \neq 1$$

26.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-\frac{1}{3}e_1+\frac{4}{3}u_1$	$\frac{1}{3}u_2$
u_2	0	0	0	$\frac{2}{3}u_2$	$-\frac{1}{2}e_1+\frac{3}{2}u_1$
u_3	$-u_1$	$\frac{1}{3}e_1-\frac{4}{3}u_1$	$-\frac{2}{3}u_2$	0	$e_1-\frac{1}{3}u_4$
u_4	$-u_2$	$-\frac{1}{3}u_2$	$\frac{1}{2}e_1-\frac{3}{2}u_1$	$\frac{1}{3}u_4-e_1$	0

27.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-\frac{1}{3}e_1+\frac{4}{3}u_1$	$\frac{1}{3}u_2$
u_2	0	0	0	$\frac{2}{3}u_2$	$\frac{1}{2}e_1-\frac{3}{2}u_1$
u_3	$-u_1$	$\frac{1}{3}e_1-\frac{4}{3}u_1$	$-\frac{2}{3}u_2$	0	$e_1-\frac{1}{3}u_4$
u_4	$-u_2$	$-\frac{1}{3}u_2$	$-\frac{1}{2}e_1+\frac{3}{2}u_1$	$\frac{1}{3}u_4-e_1$	0

28.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$2u_1$	$2u_2$
u_2	0	0	0	u_2	$e_1-\frac{1}{2}u_1$
u_3	$-u_1$	$-2u_1$	$-u_2$	0	u_4
u_4	$-u_2$	$-2u_2$	$\frac{1}{2}u_1-e_1$	$-u_4$	0

29.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$2u_1$	$2u_2$
u_2	0	0	0	u_2	$-e_1+\frac{1}{2}u_1$
u_3	$-u_1$	$-2u_1$	$-u_2$	0	u_4
u_4	$-u_2$	$-2u_2$	$e_1-\frac{1}{2}u_1$	$-u_4$	0

30.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	x	y
u_2	0	0	0	y	z
u_3	$-u_1$	$-x$	$-y$	0	0
u_4	$-u_2$	$-y$	$-z$	0	0,

where

$$
\begin{aligned}
x &= \frac{\lambda\mu(\lambda-1)}{\lambda+\mu-\lambda\mu}e_1 + \frac{\lambda^2+\mu-\lambda^2\mu}{\lambda+\mu-\lambda\mu}u_1 + \frac{\lambda(1-\lambda)}{\lambda+\mu-\lambda\mu}u_2,\\
y &= -\frac{\lambda\mu}{\lambda+\mu-\lambda\mu}e_1 + \frac{\mu}{\lambda+\mu-\lambda\mu}u_1 + \frac{\lambda}{\lambda+\mu-\lambda\mu}u_2,\\
z &= \frac{\lambda\mu(\mu-1)}{\lambda+\mu-\lambda\mu}e_1 + \frac{\mu(1-\mu)}{\lambda+\mu-\lambda\mu}u_1 + \frac{\lambda+\mu^2-\mu^2\lambda}{\lambda+\mu-\lambda\mu}u_2,
\end{aligned}
$$

$$\lambda+\mu-\lambda\mu\neq 0, \quad -1\leqslant\mu\leqslant\lambda, \quad \lambda\mu>0$$

31.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	0
u_3	$-u_1$	0	0	0	e_1
u_4	$-u_2$	0	0	$-e_1$	0

32.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	0
u_3	$-u_1$	0	0	0	0
u_4	$-u_2$	0	0	0	0

Proof. Consider the trivial pair 1.3.25.

1. The group of automophisms of the pair 1.3.25 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & ab_3 & ac_3 & b_1 & c_1 \\ 0 & ab_4 & ac_4 & b_2 & c_2 \\ 0 & 0 & 0 & b_3 & c_3 \\ 0 & 0 & 0 & b_4 & c_4 \end{pmatrix} \middle| \begin{array}{c} a \in \mathbb{C}^*, \\ b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4 \in \mathbb{C}, \\ b_3c_4 - c_3b_4 \neq 0 \end{array} \right\}.$$

2. The set of anti–involutions of the pair 1.3.25 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \alpha\beta_3 & \alpha\gamma_3 & \beta_1 & \gamma_1 \\ 0 & \alpha\beta_4 & \alpha\gamma_4 & \beta_2 & \gamma_2 \\ 0 & 0 & 0 & \beta_3 & \gamma_3 \\ 0 & 0 & 0 & \beta_4 & \gamma_4 \end{pmatrix} \middle| \begin{array}{c} I\overline{I} = E,\ \alpha \in \mathbb{C}^*, \\ \beta_1, \beta_2, \beta_3, \beta_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{C}, \\ \beta_3\gamma_4 - \gamma_3\beta_4 \neq 0 \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 1.3.25 is conjugate to one and only one:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It follows that any real form of linear Lie algebra 1.3 is conjugate to one and only one Lie algebra: 1.3^1.

Theorem 1.4. *Any real form of the linear Lie algebra* 1.4 *is conjugate to one and only one linear Lie algebra:*

$$1.4^1 \quad \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 1.4^1 *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	e_1
u_1	0	0	u_1	u_2	u_1
u_2	$-u_1$	$-u_1$	0	u_3	0
u_3	$-u_2$	$-u_2$	$-u_3$	0	$-u_3$
u_4	$-e_1$	$-u_1$	0	u_3	0

2.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	e_1
u_1	0	0	0	0	pu_1
u_2	$-u_1$	0	0	0	$(p-1)u_2$
u_3	$-u_2$	0	0	0	$(p-2)u_3$
u_4	$-e_1$	$-pu_1$	$(1-p)u_2$	$(2-p)u_3$	0

3.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	e_1
u_1	0	0	0	0	$2u_1$
u_2	$-u_1$	0	0	e_1	u_2
u_3	$-u_2$	0	$-e_1$	0	0
u_4	$-e_1$	$-2u_1$	$-u_2$	0	0

4.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	e_1
u_1	0	0	0	0	$2u_1$
u_2	$-u_1$	0	0	$-e_1$	u_2
u_3	$-u_2$	0	e_1	0	0
u_4	$-e_1$	$-2u_1$	$-u_2$	0	0

5.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	u_1	u_2	0
u_2	$-u_1$	$-u_1$	0	u_3	0
u_3	$-u_2$	$-u_2$	$-u_3$	0	0
u_4	0	0	0	0	0

6.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	u_1
u_2	$-u_1$	0	0	0	u_2
u_3	$-u_2$	0	0	0	u_1+u_3
u_4	0	$-u_1$	$-u_2$	$-u_1-u_3$	0

7.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	u_1
u_2	$-u_1$	0	0	0	u_2
u_3	$-u_2$	0	0	0	$-u_1+u_3$
u_4	0	$-u_1$	$-u_2$	u_1-u_3	0

8.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	u_1
u_2	$-u_1$	0	0	0	u_2
u_3	$-u_2$	0	0	0	u_3
u_4	0	$-u_1$	$-u_2$	$-u_3$	0

9.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	$re_1+u_2+u_4$	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2-u_4$	0	pu_4
u_4	0	0	0	$-pu_4$	0

10.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	re_1+u_2	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2$	0	pu_4
u_4	0	0	0	$-pu_4$	0

11.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	$re_1+u_2+u_4$	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2-u_4$	0	u_1-u_4
u_4	0	0	0	u_4-u_1	0

12.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	re_1+u_2	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2$	0	u_1-u_4
u_4	0	0	0	u_4-u_1	0

13.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	re_1+u_4	0
u_3	$-u_2$	0	$-re_1-u_4$	0	u_4
u_4	0	0	0	$-u_4$	0

14.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	re_1	0
u_3	$-u_2$	0	$-re_1$	0	u_4
u_4	0	0	0	$-u_4$	0

15.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1+u_4	0
u_3	$-u_2$	0	$-e_1-u_4$	0	u_1
u_4	0	0	0	$-u_1$	0

16.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	$-e_1+u_4$	0
u_3	$-u_2$	0	e_1-u_4	0	u_1
u_4	0	0	0	$-u_1$	0

17.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	u_4	0
u_3	$-u_2$	0	$-u_4$	0	u_1
u_4	0	0	0	$-u_1$	0

18.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1+u_4	0
u_3	$-u_2$	0	$-e_1-u_4$	0	0
u_4	0	0	0	0	0

19.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	$-e_1+u_4$	0
u_3	$-u_2$	0	e_1-u_4	0	0
u_4	0	0	0	0	0

20.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	u_4	0
u_3	$-u_2$	0	$-u_4$	0	0
u_4	0	0	0	0	0

21.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1	0
u_3	$-u_2$	0	$-e_1$	0	u_1
u_4	0	0	0	$-u_1$	0

22.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	$-e_1$	0
u_3	$-u_2$	0	e_1	0	u_1
u_4	0	0	0	$-u_1$	0

23.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	0
u_3	$-u_2$	0	0	0	u_1
u_4	0	0	0	$-u_1$	0

24.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1	0
u_3	$-u_2$	0	$-e_1$	0	0
u_4	0	0	0	0	0

25.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	$-e_1$	0
u_3	$-u_2$	0	e_1	0	0
u_4	0	0	0	0	0

26.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	0
u_3	$-u_2$	0	0	0	0
u_4	0	0	0	0	0

Proof. Consider the trivial pair 1.4.20.

1. The group of automorphisms of the pair 1.4.20 has the form:

$$\mathcal{A}=\left\{\left.\begin{pmatrix} a & 0 & 0 & 0 & 0\\ 0 & a^2b_3 & ab_2 & b_1 & c\\ 0 & 0 & ab_3 & b_2 & 0\\ 0 & 0 & 0 & b_3 & 0\\ 0 & 0 & 0 & b_4 & d\end{pmatrix}\right| a,b_3,d\in\mathbb{C}^*,b_1,b_2,b_4,c\in\mathbb{C}\right\}.$$

2. The set of anti–involutions of the pair 1.4.20 has the form:

$$\mathcal{I}=\left\{\left.I=\begin{pmatrix} \alpha & 0 & 0 & 0 & 0\\ 0 & \alpha^2\beta_3 & \alpha\beta_2 & \beta_1 & \gamma\\ 0 & 0 & \alpha\beta_3 & \beta_2 & 0\\ 0 & 0 & 0 & \beta_3 & 0\\ 0 & 0 & 0 & \beta_4 & \delta\end{pmatrix}\right| \begin{array}{c} I\overline{I}=E,\ \alpha,\beta_3,\delta\in\mathbb{C}^*,\\ \beta_1,\beta_2,\beta_4,\gamma\in\mathbb{C}\end{array}\right\}.$$

3. Let $A\in\mathcal{A}$, $I\in\mathcal{I}$. Using the operation $I\mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 1.4.20 is conjugate to one and only one:

$$I_1=\begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}.$$

It follows that any real form of linear Lie algebra 1.4 is conjugate to one and only one Lie algebra: 1.4^1.

2. Real forms of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ with solvable subalgebra $\mathfrak{g}$ Two-dimensional case

Theorem 2.1. *Any real form of the linear Lie algebra 2.1 is conjugate to one and only one of the following linear Lie algebras:*

$$2.1^1 \quad \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -y \end{pmatrix} \qquad 2.1^2 \quad \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & 0 & -y \\ 0 & 0 & -x & 0 \\ 0 & y & 0 & 0 \end{pmatrix}$$

$$2.1^3 \quad \begin{pmatrix} 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -y \\ x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \end{pmatrix} \qquad 2.1^4 \quad \begin{pmatrix} x & -y & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & -x & y \\ 0 & 0 & -y & -x \end{pmatrix}$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ of type 2.1^1 is equivalent to one and only one of the following pairs:

1.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_2$	0	0	0	e_2
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_4	0	$-e_2$	0	0

2.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_2$	0	0	0	0
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_4	0	0	0	0

3.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	0
u_3	u_3	0	0	0	0	0
u_4	0	u_4	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 2.1^2 *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_4$	0	0	0	e_2
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_2	0	$-e_2$	0	0

2.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_4$	0	0	0	$-e_2$
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_2	0	e_2	0	0

3.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_4$	0	0	0	0
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_2	0	0	0	0

4.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_4$	0	0	0	e_2
u_3	u_3	0	0	0	0	0
u_4	0	u_2	0	$-e_2$	0	0

5.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_4$	0	0	0	$-e_2$
u_3	u_3	0	0	0	0	0
u_4	0	u_2	0	e_2	0	0

6.

$[,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_4$	0	0	0	0
u_3	u_3	0	0	0	0	0
u_4	0	u_2	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ of type 2.1^3 *is equivalent to one and only one of the following pairs:*

1.

$[,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_3	0	$-u_1$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_3$	0	0	0	e_1	0
u_2	0	$-u_4$	0	0	0	e_2
u_3	u_1	0	$-e_1$	0	0	0
u_4	0	u_2	0	$-e_2$	0	0

2.

$[,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_3	0	$-u_1$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_3$	0	0	0	e_1	0
u_2	0	$-u_4$	0	0	0	$-e_2$
u_3	u_1	0	$-e_1$	0	0	0
u_4	0	u_2	0	e_2	0	0

3.

$[,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_3	0	$-u_1$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_3$	0	0	0	$-e_1$	0
u_2	0	$-u_4$	0	0	0	$-e_2$
u_3	u_1	0	e_1	0	0	0
u_4	0	u_2	0	e_2	0	0

4.

$[,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_3	0	$-u_1$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_3$	0	0	0	e_1	0
u_2	0	$-u_4$	0	0	0	0
u_3	u_1	0	$-e_1$	0	0	0
u_4	0	u_2	0	0	0	0

5.

$[,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_3	0	$-u_1$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_3$	0	0	0	$-e_1$	0
u_2	0	$-u_4$	0	0	0	0
u_3	u_1	0	e_1	0	0	0
u_4	0	u_2	0	0	0	0

6.

$[,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_3	0	$-u_1$	0
e_2	0	0	0	u_4	0	$-u_2$
u_1	$-u_3$	0	0	0	0	0
u_2	0	$-u_4$	0	0	0	0
u_3	u_1	0	0	0	0	0
u_4	0	u_2	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 2.1^4 *is equivalent to one and only one of the following pairs:*

1.

$[,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	u_2	$-u_1$	$-u_4$	u_3
u_1	$-u_1$	$-u_2$	0	0	e_1	e_2
u_2	$-u_2$	u_1	0	0	e_2	$-e_1$
u_3	u_3	u_4	$-e_1$	$-e_2$	0	0
u_4	u_4	$-u_3$	$-e_2$	e_1	0	0

2.

$[,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	u_2	$-u_1$	$-u_4$	u_3
u_1	$-u_1$	$-u_2$	0	0	0	0
u_2	$-u_2$	u_1	0	0	0	0
u_3	u_3	u_4	0	0	0	0
u_4	u_4	$-u_3$	0	0	0	0

The proof of the Theorem follows from Proposition 2.1.1–2.1.3.

Proposition 2.1.1. *Any real form of the pair* 2.1.1 *is equivalent to one and only one of the following pairs:*

$$2.1^1.1, \quad 2.1^2.1, \quad 2.1^2.2, \quad 2.1^3.1, \quad 2.1^3.2, \quad 2.1^3.3, \quad 2.1^4.1.$$

Proof. 1. The group of automorphisms of the pair 2.1.1 has the form:

$$\mathcal{A}=\left\{\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&a&0&0&0\\0&0&0&b&0&0\\0&0&0&0&\frac{1}{a}&0\\0&0&0&0&0&\frac{1}{b}\end{pmatrix},\begin{pmatrix}1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&a&0&0&0\\0&0&0&0&0&\frac{1}{b}\\0&0&0&0&\frac{1}{a}&0\\0&0&0&b&0&0\end{pmatrix},\right.$$
$$\begin{pmatrix}-1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&\frac{1}{a}&0\\0&0&0&b&0&0\\0&0&a&0&0&0\\0&0&0&0&0&\frac{1}{b}\end{pmatrix},\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&\frac{1}{a}&0\\0&0&0&0&0&\frac{1}{b}\\0&0&a&0&0&0\\0&0&0&b&0&0\end{pmatrix},$$
$$\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&b&0&0\\0&0&a&0&0&0\\0&0&0&0&0&\frac{1}{b}\\0&0&0&0&\frac{1}{a}&0\end{pmatrix},\begin{pmatrix}0&-1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&\frac{1}{b}\\0&0&a&0&0&0\\0&0&0&b&0&0\\0&0&0&0&\frac{1}{a}&0\end{pmatrix},$$
$$\left.\begin{pmatrix}0&1&0&0&0&0\\-1&0&0&0&0&0\\0&0&0&b&0&0\\0&0&0&0&\frac{1}{a}&0\\0&0&0&0&0&\frac{1}{b}\\0&0&a&0&0&0\end{pmatrix},\begin{pmatrix}0&-1&0&0&0&0\\-1&0&0&0&0&0\\0&0&0&0&0&\frac{1}{b}\\0&0&0&0&\frac{1}{a}&0\\0&0&0&b&0&0\\0&0&a&0&0&0\end{pmatrix}\;\middle|\; a,b\in\mathbb{C}^*\right\}.$$

2. The set of anti–involutions of the pair 2.1.1 has the form:

$$\mathcal{I}=\left\{I\in\left\{\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&\alpha&0&0&0\\0&0&0&\beta&0&0\\0&0&0&0&\frac{1}{\alpha}&0\\0&0&0&0&0&\frac{1}{\beta}\end{pmatrix},\begin{pmatrix}1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&\alpha&0&0&0\\0&0&0&0&0&\frac{1}{\beta}\\0&0&0&0&\frac{1}{\alpha}&0\\0&0&0&\beta&0&0\end{pmatrix},\right.\right.$$
$$\begin{pmatrix}-1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&\frac{1}{\alpha}&0\\0&0&0&\beta&0&0\\0&0&\alpha&0&0&0\\0&0&0&0&0&\frac{1}{\beta}\end{pmatrix},\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&\frac{1}{\alpha}&0\\0&0&0&0&0&\frac{1}{\beta}\\0&0&\alpha&0&0&0\\0&0&0&\beta&0&0\end{pmatrix},$$
$$\left.\left.\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&\beta&0&0\\0&0&\alpha&0&0&0\\0&0&0&0&0&\frac{1}{\beta}\\0&0&0&0&\frac{1}{\alpha}&0\end{pmatrix},\begin{pmatrix}0&-1&0&0&0&0\\-1&0&0&0&0&0\\0&0&0&0&0&\frac{1}{\beta}\\0&0&0&0&\frac{1}{\alpha}&0\\0&0&0&\beta&0&0\\0&0&\alpha&0&0&0\end{pmatrix}\right\}\;\middle|\;\begin{matrix}I\overline{I}=E,\\ \alpha,\beta\in\mathbb{C}^*\end{matrix}\right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.1.1 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad I_4 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$I_5 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad I_6 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$I_7 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, b, x, y, z, t)$, $I_2(X) = (\overline{a}, -\overline{b}, \overline{x}, -\overline{t}, \overline{z}, -\overline{y})$, $I_2(X) = X$. It follows that $X = (a, b, x, y, z, -\overline{y})$, where $y \in \mathbb{C}$, $b \in \mathbb{R}i$, $a, x, z \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= e_1, & e_2' &= ie_2, \\ u_1' &= u_1, & u_2' &= \frac{1}{\sqrt{2}}u_2 - \frac{1}{\sqrt{2}}u_4, \\ u_3' &= u_3, & u_4' &= \frac{i}{\sqrt{2}}u_2 + \frac{i}{\sqrt{2}}u_4. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $2.1^2.1$.

5. Find the set of fixed points of the mapping I_3.

Let $X = (a, b, x, y, z, t)$, $I_3(X) = (\overline{a}, -\overline{b}, \overline{x}, \overline{t}, \overline{z}, \overline{y})$, $I_3(X) = X$. It follows that $X = (a, b, x, y, z, \overline{y})$, where $y \in \mathbb{C}$, $b \in \mathbb{R}i$, $a, x, z \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$
\begin{aligned}
e_1' &= e_1, \quad e_2' = ie_2,\\
u_1' &= u_1, \quad u_2' = \frac{1}{\sqrt{2}}u_2 + \frac{1}{\sqrt{2}}u_4,\\
u_3' &= u_3, \quad u_4' = \frac{i}{\sqrt{2}}u_2 - \frac{i}{\sqrt{2}}u_4.
\end{aligned}
$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_3 has the form $2.1^2.2$.

6. Find the set of fixed points of the mapping I_4.

Let $X = (a, b, x, y, z, t)$, $I_4(X) = (-\overline{a}, -\overline{b}, -\overline{z}, -\overline{t}, -\overline{x}, -\overline{y})$, $I_4(X) = X$. It follows that $X = (a, b, x, y, -\overline{x}, -\overline{y})$, where $x, y \in \mathbb{C}$, $a, b \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$
\begin{aligned}
e_1' &= ie_1, \quad e_2' = ie_2,\\
u_1' &= \frac{1}{\sqrt{2}}u_1 - \frac{1}{\sqrt{2}}u_3,\\
u_2' &= \frac{1}{\sqrt{2}}u_2 - \frac{1}{\sqrt{2}}u_4,\\
u_3' &= \frac{i}{\sqrt{2}}u_1 + \frac{i}{\sqrt{2}}u_3,\\
u_4' &= \frac{i}{\sqrt{2}}u_2 + \frac{i}{\sqrt{2}}u_4.
\end{aligned}
$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_4 has the form $2.1^3.1$.

7. Find the set of fixed points of the mapping I_5.

Let $X = (a, b, x, y, z, t)$, $I_5(X) = (-\overline{a}, -\overline{b}, -\overline{z}, \overline{t}, -\overline{x}, \overline{y})$, $I_5(X) = X$. It follows that $X = (a, b, x, y, -\overline{x}, \overline{y})$, where $x, y \in \mathbb{C}$, $a, b \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$
\begin{aligned}
e_1' &= ie_1, \quad e_2' = ie_2,\\
u_1' &= \frac{1}{\sqrt{2}}u_1 - \frac{1}{\sqrt{2}}u_3,\\
u_2' &= \frac{1}{\sqrt{2}}u_2 + \frac{1}{\sqrt{2}}u_4,\\
u_3' &= \frac{i}{\sqrt{2}}u_1 + \frac{i}{\sqrt{2}}u_3,\\
u_4' &= \frac{i}{\sqrt{2}}u_2 - \frac{i}{\sqrt{2}}u_4.
\end{aligned}
$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_5 has the form $2.1^3.2$.

8. Find the set of fixed points of the mapping I_6.

Let $X = (a, b, x, y, z, t)$, $I_6(X) = (-\overline{a}, -\overline{b}, \overline{z}, \overline{t}, \overline{x}, \overline{y})$, $I_6(X) = X$. It follows that $X = (a, b, x, y, \overline{x}, \overline{y})$, where $x, y \in \mathbb{C}$, $a, b \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= ie_1, \quad e_2' = ie_2, \\ u_1' &= \frac{1}{\sqrt{2}}u_1 + \frac{1}{\sqrt{2}}u_3, \\ u_2' &= \frac{1}{\sqrt{2}}u_2 + \frac{1}{\sqrt{2}}u_4, \\ u_3' &= \frac{i}{\sqrt{2}}u_1 - \frac{i}{\sqrt{2}}u_3, \\ u_4' &= \frac{i}{\sqrt{2}}u_2 - \frac{i}{\sqrt{2}}u_4. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_6 has the form $2.1^3.3$.

9. Find the set of fixed points of the mapping I_7.

Let $X = (a, b, x, y, z, t)$, $I_7(X) = (\overline{b}, \overline{a}, \overline{y}, \overline{x}, \overline{t}, \overline{z})$, $I_7(X) = X$. It follows that $X = (a, \overline{a}, x, \overline{x}, z, \overline{z})$, where $a, x, z \in \mathbb{C}$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= e_1 + e_2, \quad e_2' = ie_1 - ie_2, \\ u_1' &= u_1 + u_2, \quad u_2' = iu_1 - iu_2, \\ u_3' &= u_3 + u_4, \quad u_4' = iu_3 - iu_4. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_7 has the form $2.1^4.1$.

Proposition 2.1.2. *Any real form of the pair* 2.1.2 *is equivalent to one and only one of the following pairs:*

$$2.1^1.2, \quad 2.1^2.3, \quad 2.1^2.4, \quad 2.1^2.5, \quad 2.1^3.4, \quad 2.1^3.5.$$

Proof. 1. The group of automorphisms of the pair 2.1.2 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & b & 0 & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \end{pmatrix} \middle| \; a, b, c \in \mathbb{C}^* \right\}.$$

2. The set of anti–involutions of the pair 2.1.2 has the form:

$$\mathcal{I}=\left\{ I\in\left\{\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&\alpha&0&0&0\\0&0&0&\beta&0&0\\0&0&0&0&\frac{1}{\alpha}&0\\0&0&0&0&0&\gamma\end{pmatrix},\begin{pmatrix}1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&\alpha&0&0&0\\0&0&0&0&0&\gamma\\0&0&0&0&\frac{1}{\alpha}&0\\0&0&0&\beta&0&0\end{pmatrix},\right.\right.$$
$$\left.\left.\begin{pmatrix}-1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&\frac{1}{\alpha}&0\\0&0&0&\beta&0&0\\0&0&\alpha&0&0&0\\0&0&0&0&0&\gamma\end{pmatrix},\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&\frac{1}{\alpha}&0\\0&0&0&0&0&\gamma\\0&0&\alpha&0&0&0\\0&0&0&\beta&0&0\end{pmatrix}\right\}\ \middle|\ \begin{array}{c}I\overline{I}=E,\\ \alpha,\beta,\gamma\in\mathbb{C}^*\end{array}\right\}.$$

3. Let $A\in\mathcal{A}$, $I\in\mathcal{I}$. Using the operation $I\mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.1.2 is conjugate to one and only one of the following:

$$I_1=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},\quad I_2=\begin{pmatrix}1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\end{pmatrix},$$

$$I_3=\begin{pmatrix}-1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&1&0&0\\0&0&-1&0&0&0\\0&0&0&0&0&1\end{pmatrix},\quad I_4=\begin{pmatrix}-1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\end{pmatrix},$$

$$I_5=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&1\\0&0&-1&0&0&0\\0&0&0&1&0&0\end{pmatrix},\quad I_6=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&1&0&0&0\\0&0&0&1&0&0\end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X=(a,b,x,y,z,t)$, $I_2(X)=(\overline{a},-\overline{b},\overline{x},\overline{t},\overline{z},\overline{y})$, $I_2(X)=X$. It follows that $X=(a,b,x,y,z,\overline{y})$, where $y\in\mathbb{C}$, $b\in\mathbb{R}i$, $a,x,z\in\mathbb{R}$.

Consider the following basis of the set of fixed points:

$$\begin{aligned}e_1'&=e_1, & e_2'&=ie_2,\\ u_1'&=u_1, & u_2'&=u_2+u_4,\\ u_3'&=u_3, & u_4'&=iu_2-iu_4.\end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $2.1^2.3$.

5. Find the set of fixed points of the mapping I_3.

Let $X = (a, b, x, y, z, t)$, $I_3(X) = (-\overline{a}, \overline{b}, -\overline{z}, \overline{y}, -\overline{x}, \overline{t})$, $I_3(X) = X$. It follows that $X = (a, b, x, y, -\overline{x}, t)$, where $x \in \mathbb{C}$, $a \in \mathbb{R}i$, $b, y, t \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$\begin{aligned}
&e_1' = e_2, \quad e_2' = ie_1,\\
&u_1' = u_2, \quad u_2' = \frac{1}{\sqrt{2}}u_1 - \frac{1}{\sqrt{2}}u_3,\\
&u_3' = u_4, \quad u_4' = \frac{i}{\sqrt{2}}u_1 + \frac{i}{\sqrt{2}}u_3.
\end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_3 has the form $2.1^2.4$.

6. Find the set of fixed points of the mapping I_4.

Let $X = (a, b, x, y, z, t)$, $I_4(X) = (-\overline{a}, \overline{b}, \overline{z}, \overline{y}, \overline{x}, \overline{t})$, $I_4(X) = X$. It follows that $X = (a, b, x, y, \overline{x}, t)$, where $x \in \mathbb{C}$, $a \in \mathbb{R}i$, $b, y, t \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$\begin{aligned}
&e_1' = e_2, \quad e_2' = ie_1,\\
&u_1' = u_2, \quad u_2' = \frac{1}{\sqrt{2}}u_1 + \frac{1}{\sqrt{2}}u_3,\\
&u_3' = u_4, \quad u_4' = \frac{i}{\sqrt{2}}u_1 - \frac{i}{\sqrt{2}}u_3.
\end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_4 has the form $2.1^2.5$.

7. Find the set of fixed points of the mapping I_5.

Let $X = (a, b, x, y, z, t)$, $I_5(X) = (-\overline{a}, -\overline{b}, -\overline{z}, \overline{t}, -\overline{x}, \overline{y})$, $I_5(X) = X$. It follows that $X = (a, b, x, y, -\overline{x}, \overline{y})$, where $x, y \in \mathbb{C}$, $a, b \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$\begin{aligned}
&e_1' = ie_1, \quad e_2' = ie_2,\\
&u_1' = \frac{1}{\sqrt{2}}u_1 - \frac{1}{\sqrt{2}}u_3,\\
&u_2' = \frac{1}{\sqrt{2}}u_2 + \frac{1}{\sqrt{2}}u_4,\\
&u_3' = \frac{i}{\sqrt{2}}u_1 + \frac{i}{\sqrt{2}}u_3,\\
&u_4' = \frac{i}{\sqrt{2}}u_2 - \frac{i}{\sqrt{2}}u_4.
\end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_5 has the form $2.1^3.4$.

8. Find the set of fixed points of the mapping I_6.

Let $X = (a, b, x, y, z, t)$, $I_6(X) = (-\overline{a}, -\overline{b}, \overline{z}, \overline{t}, \overline{x}, \overline{y})$, $I_6(X) = X$. It follows that $X = (a, b, x, y, \overline{x}, \overline{y})$, where $x, y \in \mathbb{C}$, $a, b \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= ie_1, \quad e_2' = ie_2, \\ u_1' &= \frac{1}{\sqrt{2}}u_1 + \frac{1}{\sqrt{2}}u_3, \\ u_2' &= \frac{1}{\sqrt{2}}u_2 + \frac{1}{\sqrt{2}}u_4, \\ u_3' &= \frac{i}{\sqrt{2}}u_1 - \frac{i}{\sqrt{2}}u_3, \\ u_4' &= \frac{i}{\sqrt{2}}u_2 - \frac{i}{\sqrt{2}}u_4. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_6 has the form $2.1^3.5$.

Proposition 2.1.3. *Any real form of the pair* 2.1.3 *is equivalent to one and only one of the following pairs:*

$$2.1^1.3, \quad 2.1^2.6, \quad 2.1^3.6, \quad 2.1^4.2.$$

Proof. 1. The group of automorphisms of the pair 2.1.3 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&a&0&0&0\\ 0&0&0&b&0&0\\ 0&0&0&0&c&0\\ 0&0&0&0&0&d \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&a&0&0&0\\ 0&0&0&0&0&d\\ 0&0&0&0&c&0\\ 0&0&0&b&0&0 \end{pmatrix}, \right.$$
$$\begin{pmatrix} -1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&c&0\\ 0&0&0&b&0&0\\ 0&0&a&0&0&0\\ 0&0&0&0&0&d \end{pmatrix}, \begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&0&0&c&0\\ 0&0&0&0&0&d\\ 0&0&a&0&0&0\\ 0&0&0&b&0&0 \end{pmatrix},$$
$$\begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&b&0&0\\ 0&0&a&0&0&0\\ 0&0&0&0&0&d\\ 0&0&0&0&c&0 \end{pmatrix}, \begin{pmatrix} 0&-1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&d\\ 0&0&a&0&0&0\\ 0&0&0&b&0&0\\ 0&0&0&0&c&0 \end{pmatrix},$$
$$\left. \begin{pmatrix} 0&1&0&0&0&0\\ -1&0&0&0&0&0\\ 0&0&0&b&0&0\\ 0&0&0&0&c&0\\ 0&0&0&0&0&d\\ 0&0&a&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&-1&0&0&0&0\\ -1&0&0&0&0&0\\ 0&0&0&0&0&d\\ 0&0&0&0&c&0\\ 0&0&0&b&0&0\\ 0&0&a&0&0&0 \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{C}^* \right\}.$$

2. The set of anti–involutions of the pair 2.1.3 has the form:

$$\mathcal{I} = \left\{ I \in \left\{ \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&\alpha&0&0&0\\ 0&0&0&\beta&0&0\\ 0&0&0&0&\gamma&0\\ 0&0&0&0&0&\delta \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&\alpha&0&0&0\\ 0&0&0&0&0&\delta\\ 0&0&0&0&\gamma&0\\ 0&0&0&\beta&0&0 \end{pmatrix}, \begin{pmatrix} -1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&\gamma&0\\ 0&0&0&\beta&0&0\\ 0&0&\alpha&0&0&0\\ 0&0&0&0&0&\delta \end{pmatrix}, \begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&0&0&\gamma&0\\ 0&0&0&0&0&\delta\\ 0&0&\alpha&0&0&0\\ 0&0&0&\beta&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&\beta&0&0\\ 0&0&\alpha&0&0&0\\ 0&0&0&0&0&\delta\\ 0&0&0&0&\gamma&0 \end{pmatrix}, \begin{pmatrix} 0&-1&0&0&0&0\\ -1&0&0&0&0&0\\ 0&0&0&0&0&\delta\\ 0&0&0&0&\gamma&0\\ 0&0&0&\beta&0&0\\ 0&0&\alpha&0&0&0 \end{pmatrix} \right\} \;\middle|\; \begin{array}{c} I\overline{I} = E, \\ \alpha,\beta,\gamma,\delta \in \mathbb{C}^* \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.1.3 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0 \end{pmatrix},$$

$$I_3 = \begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}, \qquad I_4 = \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a,b,x,y,z,t)$, $I_2(X) = (\overline{a}, -\overline{b}, \overline{x}, \overline{t}, \overline{z}, \overline{y})$, $I_2(X) = X$. It follows that $X = (a,b,x,y,z,\overline{y})$, where $y \in \mathbb{C}$, $b \in \mathbb{R}i$, $a,x,z \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$e_1' = e_1, \quad e_2' = ie_2,$$

$$u_1' = u_1, \quad u_2' = \frac{1}{\sqrt{2}}u_2 + \frac{1}{\sqrt{2}}u_4,$$

$$u_3' = u_3, \quad u_4' = \frac{i}{\sqrt{2}}u_2 - \frac{i}{\sqrt{2}}u_4.$$

In this basis the multiplication table of the real form corresponding to the anti-involution I_2 has the form $2.1^2.6$.

5. Find the set of fixed points of the mapping I_3.

Let $X = (a, b, x, y, z, t)$, $I_3(X) = (-\overline{a}, -\overline{b}, \overline{z}, \overline{t}, \overline{x}, \overline{y})$, $I_3(X) = X$. It follows that $X = (a, b, x, y, \overline{x}, \overline{y})$, where $x, y \in \mathbb{C}$, $a, b \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= ie_1, \quad e_2' = ie_2, \\ u_1' &= \frac{1}{\sqrt{2}}u_1 + \frac{1}{\sqrt{2}}u_3, \\ u_2' &= \frac{1}{\sqrt{2}}u_2 + \frac{1}{\sqrt{2}}u_4, \\ u_3' &= \frac{i}{\sqrt{2}}u_1 - \frac{i}{\sqrt{2}}u_3, \\ u_4' &= \frac{i}{\sqrt{2}}u_2 - \frac{i}{\sqrt{2}}u_4. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti-involution I_3 has the form $2.1^3.6$.

6. Find the set of fixed points of the mapping I_4.

Let $X = (a, b, x, y, z, t)$, $I_4(X) = (\overline{b}, \overline{a}, \overline{y}, \overline{x}, \overline{t}, \overline{z})$, $I_4(X) = X$. It follows that $X = (a, \overline{a}, x, \overline{x}, z, \overline{z})$, where $a, x, z \in \mathbb{C}$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= e_1 + e_2, & e_2' &= ie_1 - ie_2, \\ u_1' &= u_1 + u_2, & u_2' &= iu_1 - iu_2, \\ u_3' &= u_3 + u_4, & u_4' &= iu_3 - iu_4. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti-involution I_4 has the form $2.1^4.2$.

Theorem 2.2. *Any real form of the linear Lie algebra 2.2 is conjugate to one and only one of the following linear Lie algebras:*

$$2.2^1 \quad \begin{pmatrix} x & y & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & -\lambda x \end{pmatrix}, \ \lambda \in [-1, 1] \qquad 2.2^2 \quad \begin{pmatrix} 0 & -x & y & 0 \\ x & 0 & 0 & y \\ 0 & 0 & 0 & -x \\ 0 & 0 & x & 0 \end{pmatrix}$$

$$2.2^3 \quad \begin{pmatrix} -x\sin\frac{\phi}{2} & x\cos\frac{\phi}{2} & y & 0 \\ -x\cos\frac{\phi}{2} & -x\sin\frac{\phi}{2} & 0 & y \\ 0 & 0 & x\sin\frac{\phi}{2} & x\cos\frac{\phi}{2} \\ 0 & 0 & -x\cos\frac{\phi}{2} & x\sin\frac{\phi}{2} \end{pmatrix}, \ \phi \in]0, \pi[$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ of type 2.2^1 is equivalent to one and only one of the following pairs:

$\lambda = 0$

1.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	$-u_4$	$-2e_2$
u_1	$-u_1$	0	0	0	u_2	$-u_1$
u_2	0	$-u_1$	0	0	0	u_2
u_3	u_3	u_4	$-u_2$	0	0	$2u_3$
u_4	0	$2e_2$	u_1	$-u_2$	$-2u_3$	0

2.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	e_2	u_4	0
u_2	0	$-u_1$	$-e_2$	0	$(p-1)u_3$	pu_4
u_3	u_3	u_4	$-u_4$	$(1-p)u_3$	0	0
u_4	0	0	0	$-pu_4$	0	0

3.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_1$	0	0	u_3	u_4
u_3	u_3	u_4	0	$-u_3$	0	0
u_4	0	0	0	$-u_4$	0	0

$\lambda = 1$

4.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	e_2	0
u_2	$-u_2$	$-u_1$	0	0	e_1	e_2
u_3	u_3	u_4	$-e_2$	$-e_1$	0	0
u_4	u_4	0	0	$-e_2$	0	0

5.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	e_2	0
u_3	u_3	u_4	0	$-e_2$	0	0
u_4	u_4	0	0	0	0	0

$\lambda = -\frac{1}{2}$

6.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$\frac{3}{2}e_2$	u_1	$-\frac{1}{2}u_2$	$-u_3$	$\frac{1}{2}u_4$
e_2	$-\frac{3}{2}e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	u_4	0	0
u_2	$\frac{1}{2}u_2$	$-u_1$	$-u_4$	0	0	0
u_3	u_3	u_4	0	0	0	0
u_4	$-\frac{1}{2}u_4$	0	0	0	0	0

$\lambda \in [-1, 1]$

7.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$(1-\lambda)e_2$	u_1	λu_2	$-u_3$	$-\lambda u_4$
e_2	$(\lambda-1)e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_2$	$-u_1$	0	0	0	0
u_3	u_3	u_4	0	0	0	0
u_4	λu_4	0	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 2.2^2 *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_2	$-u_1$	u_4	$-u_3$
e_2	0	0	0	0	u_1	u_2
u_1	$-u_2$	0	0	0	e_2	0
u_2	u_1	0	0	0	0	e_2
u_3	$-u_4$	$-u_1$	$-e_2$	0	0	$-e_1$
u_4	u_3	$-u_2$	0	$-e_2$	e_1	0

2.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_2	$-u_1$	u_4	$-u_3$
e_2	0	0	0	0	u_1	u_2
u_1	$-u_2$	0	0	0	$-e_2$	0
u_2	u_1	0	0	0	0	$-e_2$
u_3	$-u_4$	$-u_1$	e_2	0	0	e_1
u_4	u_3	$-u_2$	0	e_2	$-e_1$	0

3.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_2	$-u_1$	u_4	$-u_3$
e_2	0	0	0	0	u_1	u_2
u_1	$-u_2$	0	0	0	0	0
u_2	u_1	0	0	0	0	0
u_3	$-u_4$	$-u_1$	0	0	0	e_2
u_4	u_3	$-u_2$	0	0	$-e_2$	0

4.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_2	$-u_1$	u_4	$-u_3$
e_2	0	0	0	0	u_1	u_2
u_1	$-u_2$	0	0	0	0	0
u_2	u_1	0	0	0	0	0
u_3	$-u_4$	$-u_1$	0	0	0	0
u_4	u_3	$-u_2$	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 2.2^3 *is equivalent to one and only one pair:*

1.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$-2\sin\frac{\phi}{2}e_2$	A	B	C	D
e_2	$2\sin\frac{\phi}{2}e_2$	0	0	0	u_1	u_2
u_1	$-A$	0	0	0	0	0
u_2	$-B$	0	0	0	0	0
u_3	$-C$	$-u_1$	0	0	0	0
u_4	$-D$	$-u_2$	0	0	0	0,

where

$$\begin{cases} A = -\sin\frac{\phi}{2}u_1 - \cos\frac{\phi}{2}u_2, \\ B = \cos\frac{\phi}{2}u_1 - \sin\frac{\phi}{2}u_2, \\ C = \sin\frac{\phi}{2}u_3 - \cos\frac{\phi}{2}u_4, \\ D = \cos\frac{\phi}{2}u_3 + \sin\frac{\phi}{2}u_4 \end{cases}$$

Proof. Let $\lambda = 0$.

Consider the trivial pair 2.2.7.

1. The group of automorphisms of the pair 2.2.7 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & bc & -ac & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & f & ad & bd \end{pmatrix} \middle| \; b, c, d \in \mathbb{C}^*, a, f \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 2.2.7 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta\gamma & -\alpha\gamma & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \varepsilon & \alpha\delta & \beta\delta \end{pmatrix} \middle| \begin{array}{c} I\overline{I} = E,\ \beta,\gamma,\delta \in \mathbb{C}^*, \\ \alpha,\varepsilon \in \mathbb{C} \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.2.7 is conjugate to one and only one:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It follows that any real form of the linear Lie algebra 2.2 ($\lambda = 0$) is conjugate to one and only one linear Lie algebra: $2.2^1(\lambda = 0)$.

If $\lambda \neq 0$ the proof of the Theorem follows from Propositions 2.2.4–2.2.7.

Proposition 2.2.4. *Any real form of the pair* 2.2.4 *is equivalent to one and only one of the following pairs:*

$$2.2^1.4, \quad 2.2^2.1, \quad 2.2^2.2.$$

Proof. 1. The group of automorphisms of the pair 2.2.4 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & ab & c & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{b} & 0 \\ 0 & 0 & 0 & 0 & d & \frac{a}{b} \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & -\frac{a}{b} \\ 0 & 0 & 0 & 0 & \frac{1}{b} & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & -ab & c & 0 & 0 \end{pmatrix} \middle| \begin{array}{c} a,b \in \mathbb{C}^*, \\ c,d \in \mathbb{C} \end{array} \right\}.$$

2. The set of anti–involutions of the pair 2.2.4 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha\beta & \gamma & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & 0 & 0 & \delta & \frac{\alpha}{\beta} \end{pmatrix}, \text{ or } \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta & -\frac{\alpha}{\beta} \\ 0 & 0 & 0 & 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & -\alpha\beta & \gamma & 0 & 0 \end{pmatrix} \middle| \begin{array}{c} I\overline{I} = E, \\ \alpha,\beta \in \mathbb{C}^*, \\ \gamma,\delta \in \mathbb{C} \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.2.4 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$I_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, b, x, y, z, t)$, $I_2(X) = (-\overline{a}, -\overline{b}, \overline{t}, \overline{z}, \overline{y}, \overline{x})$, $I_2(X) = X$. It follows that $X = (a, b, x, y, \overline{y}, \overline{x})$, where $x, y \in \mathbb{C}$, $a, b \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= ie_1, \quad e_2' = ie_2, \\ u_1' &= \frac{1}{\sqrt{2}}u_1 + \frac{1}{\sqrt{2}}u_4, \\ u_2' &= \frac{i}{\sqrt{2}}u_1 - \frac{i}{\sqrt{2}}u_4, \\ u_3' &= \frac{i}{\sqrt{2}}u_3 - \frac{i}{\sqrt{2}}u_2, \\ u_4' &= u_2 + u_3. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form 2.2^2.1.

5. Find the set of fixed points of the mapping I_3.

Let $X = (a, b, x, y, z, t)$, $I_3(X) = (-\overline{a}, -\overline{b}, -\overline{t}, -\overline{z}, -\overline{y}, -\overline{x})$, $I_3(X) = X$. It follows that $X = (a, b, x, y, -\overline{y}, -\overline{x})$, where $x, y \in \mathbb{C}$, $a, b \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= ie_1, \quad e_2' = ie_2, \\ u_1' &= \frac{1}{\sqrt{2}}u_1 - \frac{1}{\sqrt{2}}u_4, \\ u_2' &= \frac{i}{\sqrt{2}}u_1 + \frac{i}{\sqrt{2}}u_4, \\ u_3' &= -\frac{i}{\sqrt{2}}u_2 - \frac{i}{\sqrt{2}}u_3, \\ u_4' &= u_2 - u_3. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_3 has the form 2.2^2.2.

Proposition 2.2.5. *Any real form of the pair 2.2.5 is equivalent to one and only one of the following pairs:*

$$2.2^1.5, \quad 2.2^2.3.$$

Proof. 1. The group of automorphisms of the pair 2.2.5 has the form:

$$\mathcal{A}=\left\{\begin{pmatrix}1&0&0&0&0&0\\0&ab&0&0&0&0\\0&0&a^2b&c&0&0\\0&0&0&a&0&0\\0&0&0&0&b&0\\0&0&0&0&d&ab^2\end{pmatrix},\begin{pmatrix}-1&0&0&0&0&0\\0&-ab&0&0&0&0\\0&0&0&0&d&ab^2\\0&0&0&0&b&0\\0&0&0&a&0&0\\0&0&a^2b&c&0&0\end{pmatrix}\middle|\begin{array}{c}a,b\in\mathbb{C}^*,\\c,d\in\mathbb{C}\end{array}\right\}.$$

2. The set of anti–involutions of the pair 2.2.5 has the form:

$$\mathcal{I}=\left\{I=\begin{pmatrix}1&0&0&0&0&0\\0&\alpha\beta&0&0&0&0\\0&0&\alpha^2\beta&\gamma&0&0\\0&0&0&\alpha&0&0\\0&0&0&0&\beta&0\\0&0&0&0&\delta&\alpha\beta^2\end{pmatrix},\text{ or}\right.$$

$$\left.\begin{pmatrix}-1&0&0&0&0&0\\0&-\alpha\beta&0&0&0&0\\0&0&0&0&\delta&\alpha\beta^2\\0&0&0&0&\beta&0\\0&0&0&\alpha&0&0\\0&0&\alpha^2\beta&\gamma&0&0\end{pmatrix}\middle|\begin{array}{c}I\overline{I}=E,\\\alpha,\beta\in\mathbb{C}^*,\\\gamma,\delta\in\mathbb{C}\end{array}\right\}.$$

3. Let $A\in\mathcal{A}$, $I\in\mathcal{I}$. Using the operation $I\mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.2.5 is conjugate to one and only one of the following:

$$I_1=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},\qquad I_2=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X=(a,b,x,y,z,t)$, $I_2(X)=(-\overline{a},-\overline{b},\overline{t},\overline{z},\overline{y},\overline{x})$, $I_2(X)=X$. It follows that $X=(a,b,x,y,\overline{y},\overline{x})$, where $x,y\in\mathbb{C}$, $a,b\in\mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1'=ie_1,\quad e_2'=-ie_2,$$

$$u_1'=-\frac{1}{\sqrt{2}}u_1-\frac{1}{\sqrt{2}}u_4,$$

$$u_2'=-\frac{i}{\sqrt{2}}u_1+\frac{i}{\sqrt{2}}u_4,$$

$$u_3'=\frac{i}{\sqrt{2}}u_3-\frac{i}{\sqrt{2}}u_2,$$

$$u_4'=u_2+u_3.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $2.2^2.3$.

Proposition 2.2.6. *Any real form of the pair* 2.2.6 *is equivalent to one and only one pair:*

$$2.2^1.6.$$

Proof. 1. The group of automorphisms of the pair 2.2.6 has the form:

$$\mathcal{A}=\left\{\left.\begin{pmatrix}1&0&0&0&0&0\\a&b&0&0&0&0\\0&0&bc&-\frac{2ac}{3}&0&0\\0&0&0&c&0&0\\0&0&0&0&c^2&0\\0&0&0&0&\frac{2ac^2}{3}&bc^2\end{pmatrix}\right|\, b,c\in\mathbb{C}^*, a\in\mathbb{C}\right\}.$$

2. The set of anti–involutions of the pair 2.2.6 has the form:

$$\mathcal{I}=\left\{I=\left.\begin{pmatrix}1&0&0&0&0&0\\\alpha&\beta&0&0&0&0\\0&0&\beta\gamma&-\frac{2\alpha\gamma}{3}&0&0\\0&0&0&\gamma&0&0\\0&0&0&0&\gamma^2&0\\0&0&0&0&\frac{2\alpha\gamma^2}{3}&\beta\gamma^2\end{pmatrix}\right|\,\begin{matrix}I\overline{I}=E,\ \beta,\gamma\in\mathbb{C}^*,\\ \alpha\in\mathbb{C}\end{matrix}\right\}.$$

3. Let $A\in\mathcal{A}$, $I\in\mathcal{I}$. Using the operation $I\mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.2.6 is conjugate to one and only one of the following:

$$I_1=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}.$$

Proposition 2.2.7. *Any real form of the pair* 2.2.7 *is equivalent to one and only one of the following pairs:*

$$2.2^1.7\ (\operatorname{Re}\lambda\in[-1,1]\backslash\{0\},\ \operatorname{Im}\lambda=0),\quad 2.2^2.4\ (\lambda=1),$$
$$2.2^3.1\ (|\lambda|=1,\ \arg\lambda\in]0,\pi[).$$

Proof. Consider the following cases:

1°. $\lambda=1$.

1. The group of automorphisms of the pair 2.2.7 has the form:

$$\mathcal{A}=\left\{\left.\begin{pmatrix}1&0&0&0&0&0\\0&a&0&0&0&0\\0&0&ab&c&0&0\\0&0&0&b&0&0\\0&0&0&0&d&0\\0&0&0&0&f&ad\end{pmatrix},\begin{pmatrix}-1&0&0&0&0&0\\0&a&0&0&0&0\\0&0&0&0&f&-ad\\0&0&0&0&d&0\\0&0&0&b&0&0\\0&0&-ab&c&0&0\end{pmatrix}\right|\,\begin{matrix}a,b,d\in\mathbb{C}^*,\\ c,f\in\mathbb{C}\end{matrix}\right\}.$$

2. The set of anti–involutions of the pair 2.2.7 has the form:

$$\mathcal{I}=\left\{I=\begin{pmatrix}1&0&0&0&0&0\\0&\alpha&0&0&0&0\\0&0&\alpha\beta&\gamma&0&0\\0&0&0&\beta&0&0\\0&0&0&0&\delta&0\\0&0&0&0&\varepsilon&\alpha\delta\end{pmatrix},\text{ or}\right.$$

$$\left.\begin{pmatrix}-1&0&0&0&0&0\\0&\alpha&0&0&0&0\\0&0&0&0&\varepsilon&-\alpha\delta\\0&0&0&0&\delta&0\\0&0&0&\beta&0&0\\0&0&-\alpha\beta&\gamma&0&0\end{pmatrix}\;\middle|\;\begin{matrix}I\overline{I}=E,\\ \alpha,\beta,\delta\in\mathbb{C}^*,\\ \gamma,\varepsilon\in\mathbb{C}\end{matrix}\right\}.$$

3. Let $A\in\mathcal{A}$, $I\in\mathcal{I}$. Using the operation $I\mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.2.7 is conjugate to one and only one of the following:

$$I_1=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},\quad I_2=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X=(a,b,x,y,z,t)$, $I_2(X)=(-\overline{a},-\overline{b},\overline{t},\overline{z},\overline{y},\overline{x})$, $I_2(X)=X$. It follows that $X=(a,b,x,y,\overline{y},\overline{x})$, where $x,y\in\mathbb{C}$, $a,b\in\mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$\begin{aligned}
e_1'&=ie_1,\quad e_2'=ie_2,\\
u_1'&=\frac{1}{\sqrt{2}}u_1+\frac{1}{\sqrt{2}}u_4,\\
u_2'&=\frac{i}{\sqrt{2}}u_1-\frac{i}{\sqrt{2}}u_4,\\
u_3'&=\frac{i}{\sqrt{2}}u_3-\frac{i}{\sqrt{2}}u_2,\\
u_4'&=u_2+u_3.
\end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $2.2^2.4$.

2°. $\lambda=-1$.

1. The group of automorphisms of the pair 2.2.7 has the form:

$$\mathcal{A}=\left\{\begin{pmatrix}1&0&0&0&0&0\\a&b&0&0&0&0\\0&0&bc_1&-\frac{ac_1}{2}&-\frac{ad_1}{2}&-bd_1\\0&0&0&c_1&d_1&0\\0&0&0&c_2&d_2&0\\0&0&-bc_2&\frac{ac_2}{2}&\frac{ad_2}{2}&bd_2\end{pmatrix}\;\middle|\;\begin{matrix}b\in\mathbb{C}^*,\\ a,c_1,c_2,d_1,d_2\in\mathbb{C},\\ c_1d_2-d_1c_2\neq 0\end{matrix}\right\}.$$

2. The set of anti–involutions of the pair 2.2.7 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta\gamma_1 & -\frac{\alpha\gamma_1}{2} & -\frac{\alpha\delta_1}{2} & -\beta\delta_1 \\ 0 & 0 & 0 & \gamma_1 & \delta_1 & 0 \\ 0 & 0 & 0 & \gamma_2 & \delta_2 & 0 \\ 0 & 0 & -\beta\gamma_2 & \frac{\alpha\gamma_2}{2} & \frac{\alpha\delta_2}{2} & \beta\delta_2 \end{pmatrix} \,\middle|\, \begin{array}{c} I\overline{I} = E,\ \beta \in \mathbb{C}^*, \\ \alpha, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{C}, \\ \gamma_1\delta_2 - \delta_1\gamma_2 \neq 0 \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.2.7 is conjugate to one and only one:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3°. $\operatorname{Re}\lambda \in]-1,1[\setminus\{0\}$, $\operatorname{Im}\lambda = 0$.

1. The group of automorphisms of the pair 2.2.7 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & bc & -\frac{ac}{1-\lambda} & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & \frac{ad}{1-\lambda} & bd \end{pmatrix} \,\middle|\, b, c, d \in \mathbb{C}^*,\ a \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 2.2.7 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta\gamma & -\frac{\alpha\gamma}{1-\lambda} & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha\delta}{1-\lambda} & \beta\delta \end{pmatrix} \,\middle|\, \begin{array}{c} I\overline{I} = E,\ \beta, \gamma, \delta \in \mathbb{C}^*, \\ \alpha \in \mathbb{C} \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.2.7 is conjugate to one and only one:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4°. $|\lambda| = 1$, $\arg\lambda \in]0, \pi[$.

1. The group of automorphisms of the pair 2.2.7 has the form:

$$\mathcal{A} = \left\{ \left. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & bc & -\frac{ac}{1-\lambda} & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & \frac{ad}{1-\lambda} & bd \end{pmatrix} \right| b, c, d \in \mathbb{C}^*,\ a \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 2.2.7 has the form:

$$\mathcal{I} = \left\{ I = \left. \begin{pmatrix} \frac{1-\overline{\lambda}}{1-\lambda} & 0 & 0 & 0 & 0 & 0 \\ \gamma & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\gamma\delta(\lambda-1)}{2(1-Re\lambda)} & -\alpha\delta \\ 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & -\alpha\beta & \frac{\beta\gamma}{1-\overline{\lambda}} & 0 & 0 \end{pmatrix} \right| \begin{matrix} I\overline{I} = E,\ \alpha, \beta, \delta \in \mathbb{C}^*, \\ \gamma \in \mathbb{C} \end{matrix} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.2.7 is conjugate to one and only one:

$$I_1 = \begin{pmatrix} \frac{1-\overline{\lambda}}{1-\lambda} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_1.

Let $X = (a, b, x, y, z, t)$, $I_1(X) = (\frac{1-\overline{\lambda}}{1-\lambda}\overline{a}, \overline{b}, -\overline{t}, \overline{z}, \overline{y}, -\overline{x})$, $I_1(X) = X$. It follows that $X = (-iae^{-\frac{i\phi}{2}}, b, x, y, \overline{y}, -\overline{x})$, where $x, y \in \mathbb{C}$, $a, b \in \mathbb{R}$, $\phi = \arg\lambda$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= -ie^{-\frac{i\phi}{2}}e_1, \quad & e_2' &= e_2, \\ u_1' &= u_1 - u_4, \quad & u_2' &= iu_1 + iu_4, \\ u_3' &= u_2 + u_3, \quad & u_4' &= iu_2 - iu_3. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_1 has the form $2.2^3.1$.

For other values of λ there exist no anti–involution.

Theorem 2.3. *Any real form of the linear Lie algebra 2.3 is conjugate to one and only one linear Lie algebra:*

$$2.3^1 \quad \begin{pmatrix} x & y & 0 & x \\ 0 & -x & -x & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & x \end{pmatrix}$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 2.3^1 *is equivalent to one and only one pair:*

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	u_1	$-u_2$	$-u_2-u_3$	u_1+u_4
e_2	$-2e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0
u_3	u_2+u_3	u_4	0	0	0	0
u_4	$-u_1-u_4$	0	0	0	0	0

Proof. Consider the trivial pair 2.3.1.

1. The group of automorphisms of the pair 2.3.1 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & bc & -\frac{ac}{2} & -\frac{ad}{2} & -bd \\ 0 & 0 & 0 & c & d & 0 \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & \frac{ac}{2} & bc \end{pmatrix} \;\middle|\; b, c \in \mathbb{C}^*,\, a, d \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 2.3.1 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta\gamma & -\frac{\alpha\gamma}{2} & -\frac{\alpha\delta}{2} & -\beta\delta \\ 0 & 0 & 0 & \gamma & \delta & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha\gamma}{2} & \beta\gamma \end{pmatrix} \;\middle|\; \begin{array}{c} I\bar{I} = E,\, \beta, \gamma \in \mathbb{C}^*, \\ \alpha, \delta \in \mathbb{C} \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\bar{A}^{-1}$ we obtain that any anti–involution of the pair 2.3.1 is conjugate to one and only one:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It follows that any real form of the linear Lie algebra 2.3 is conjugate to one and only one linear Lie algebra: 2.3^1.

Theorem 2.4. *Any real form of the linear Lie algebra* 2.4 *is conjugate to one and only one linear Lie algebra:*

$$2.4^1 \quad \begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 2.4^1 *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	u_2	0
u_1	$-u_1$	0	0	u_1	u_2	0
u_2	0	$-u_1$	$-u_1$	0	u_3	0
u_3	u_3	$-u_2$	$-u_2$	$-u_3$	0	0
u_4	0	0	0	0	0	0

2.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	u_2	0
u_1	$-u_1$	0	0	0	0	u_1
u_2	0	$-u_1$	0	0	0	u_2
u_3	u_3	$-u_2$	0	0	0	u_3
u_4	0	0	$-u_1$	$-u_2$	$-u_3$	0

3.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	u_2	0
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_1$	0	0	0	0
u_3	u_3	$-u_2$	0	0	0	0
u_4	0	0	0	0	0	0

Proof. Consider the trivial pair 2.4.3.

1. The group of automorphisms of the pair 2.4.3 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & b^2c & -abc & \frac{a^2c}{2} & 0 \\ 0 & 0 & 0 & bc & -ac & 0 \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & d \end{pmatrix} \middle| \; b, c, d \in \mathbb{C}^*,\ a \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 2.4.3 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta^2\gamma & -\alpha\beta\gamma & \frac{\alpha^2\gamma}{2} & 0 \\ 0 & 0 & 0 & \beta\gamma & -\alpha\gamma & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta \end{pmatrix} \middle| \; \begin{array}{c} I\bar{I} = E,\ \beta, \gamma, \delta \in \mathbb{C}^*, \\ \alpha \in \mathbb{C} \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\bar{A}^{-1}$ we obtain that any

anti–involution of the pair 2.4.3 is conjugate to one and only one:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It follows that any real form of the linear Lie algebra 2.4 is conjugate to one and only one linear Lie algebra: 2.4^1.

Theorem 2.5. *Any real form of the linear Lie algebra* 2.5 *is conjugate to one and only one of the following linear Lie algebras:*

$$2.5^1 \quad \begin{pmatrix} 0 & x & 0 & y \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \end{pmatrix} \qquad 2.5^2 \quad \begin{pmatrix} 0 & x & 0 & -y \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 \end{pmatrix}$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 2.5^1 *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	$-2e_1$
e_2	0	0	0	$-2e_2$	$-u_2$	u_1
u_1	0	0	0	$2e_2-u_1$	u_2+u_4	$2e_1-u_1$
u_2	$-u_1$	$2e_2$	u_1-2e_2	0	$-2u_3$	u_2-u_4
u_3	u_4	u_2	$-u_2-u_4$	$2u_3$	0	$2u_3$
u_4	$2e_1$	$-u_1$	u_1-2e_1	u_4-u_2	$-2u_3$	0

2.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	$-2e_2$	$-u_2$	u_1
u_1	0	0	0	$-u_1$	u_4	0
u_2	$-u_1$	$2e_2$	u_1	0	$-2u_3$	$-u_4$
u_3	u_4	u_2	$-u_4$	$2u_3$	0	0
u_4	0	$-u_1$	0	u_4	0	0

3.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	u_1	0
u_2	$-u_1$	0	0	0	$e_1+ge_2+(1-h)u_2$	hu_1
u_3	u_4	u_2	$-u_1$	$-e_1-ge_2+(h-1)u_2$	0	$-(g+h)e_1+ke_2-(1+h)u_4$
u_4	0	$-u_1$	0	$-hu_1$	$(g+h)e_1-ke_2+(1+h)u_4$	0

$h \geqslant 0$ (*if* $k \neq 0$), $h \in \mathbb{R}$ (*if* $k = 0$)

4.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	u_1	0
u_2	$-u_1$	0	0	0	$ge_2+(1-h)u_2$	hu_1
u_3	u_4	u_2	$-u_1$	$-ge_2+(h-1)u_2$	0	$-(g+h)e_1-(1+h)u_4$
u_4	0	$-u_1$	0	$-hu_1$	$(g+h)e_1+(1+h)u_4$	0

$$h \geqslant 0$$

5.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	$e_1+ge_2-u_2$	u_1
u_3	u_4	u_2	0	$-e_1-ge_2+u_2$	0	$-ge_1+ke_2-u_4$
u_4	0	$-u_1$	0	$-u_1$	$ge_1-ke_2+u_4$	0

6.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	ge_2-u_2	u_1
u_3	u_4	u_2	0	$-ge_2+u_2$	0	$-ge_1-u_4$
u_4	0	$-u_1$	0	$-u_1$	ge_1+u_4	0

7.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1+e_2	0
u_3	u_4	u_2	0	$-e_1-e_2$	0	$-e_1+ke_2$
u_4	0	$-u_1$	0	0	e_1-ke_2	0

8.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1-e_2	0
u_3	u_4	u_2	0	$-e_1+e_2$	0	e_1+ke_2
u_4	0	$-u_1$	0	0	$-e_1-ke_2$	0

9.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_2	0
u_3	u_4	u_2	0	$-e_2$	0	$-e_1$
u_4	0	$-u_1$	0	0	e_1	0

10.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	$-e_2$	0
u_3	u_4	u_2	0	e_2	0	e_1
u_4	0	$-u_1$	0	0	$-e_1$	0

11.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1	0
u_3	u_4	u_2	0	$-e_1$	0	e_2
u_4	0	$-u_1$	0	0	$-e_2$	0

12.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1	0
u_3	u_4	u_2	0	$-e_1$	0	$-e_2$
u_4	0	$-u_1$	0	0	e_2	0

13.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1	0
u_3	u_4	u_2	0	$-e_1$	0	0
u_4	0	$-u_1$	0	0	0	0

14.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	0	0
u_3	u_4	u_2	0	0	0	0
u_4	0	$-u_1$	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ of type 2.5^2 is equivalent to one and only one of the following pairs:

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	$-e_1+u_1$	$-u_2$	e_2
e_2	0	0	0	$-e_2$	u_4	$-e_1-u_1$
u_1	0	0	0	e_1-u_1	u_2	$-e_2$
u_2	e_1-u_1	e_2	$-e_1+u_1$	0	$-2u_3$	$-u_4$
u_3	u_2	$-u_4$	$-u_2$	$2u_3$	0	0
u_4	$-e_2$	e_1+u_1	e_2	u_4	0	0

2.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_2$	0
e_2	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	u_1	0
u_2	$-u_1$	0	0	0	A	$2ru_1$
u_3	u_2	$-u_4$	$-u_1$	$-A$	0	B
u_4	0	u_1	0	$-2ru_1$	$-B$	0 ,

where

$$\begin{cases} A = (p+s)e_1 + re_2 + u_2 - 2ru_4, \\ B = -re_1 + (p-s)e_2 - 2ru_2 - u_4, \\ r \geqslant 0,\ s \geqslant 0 \end{cases}$$

3.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_2$	0
e_2	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	$-(r+s)e_1-u_4$	u_1
u_3	u_2	$-u_4$	0	$(r+s)e_1+u_4$	0	$(s-r)e_2-u_2$
u_4	0	u_1	0	$-u_1$	$(r-s)e_2+u_2$	0

$$s \geqslant 0$$

4.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_2$	0
e_2	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	$(1+s)e_1$	0
u_3	u_2	$-u_4$	0	$-(1+s)e_1$	0	$(1-s)e_2$
u_4	0	u_1	0	0	$(s-1)e_2$	0

$$s \geqslant 0$$

5.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_2$	0
e_2	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	$-(1+s)e_1$	0
u_3	u_2	$-u_4$	0	$(1+s)e_1$	0	$(s-1)e_2$
u_4	0	u_1	0	0	$(1-s)e_2$	0

$$s \geqslant 0$$

6.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_2$	0
e_2	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_2	0
u_3	u_2	$-u_4$	0	$-e_2$	0	e_1
u_4	0	u_1	0	0	$-e_1$	0

7.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_2$	0
e_2	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	0	0
u_3	u_2	$-u_4$	0	0	0	0
u_4	0	u_1	0	0	0	0

The proof of the Theorem follows from Propositions 2.5.1–2.5.11.

Proposition 2.5.1. *Any real form of the pair* 2.5.1 *is equivalent to one and only one of the following pairs:*

$$2.5^1.1, \quad 2.5^2.1.$$

Proof. 1. The group of automorphisms of the pair 2.5.1 has the form:

$$\mathcal{A}=\left\{\begin{pmatrix} a & 0 & 0 & 0 & ac^2 & 2ac\\ 0 & a & 0 & 2ab & ab^2 & 0\\ 0 & 0 & a & -ac & -abc & -ab\\ 0 & 0 & 0 & 1 & b & 0\\ 0 & 0 & 0 & 0 & \frac{1}{a} & 0\\ 0 & 0 & 0 & 0 & c & 1 \end{pmatrix}, \begin{pmatrix} 0 & a & 0 & 2ac & ac^2 & 0\\ a & 0 & 0 & 0 & ab^2 & 2ab\\ 0 & 0 & a & -ab & -abc & -ac\\ 0 & 0 & 0 & 0 & b & 1\\ 0 & 0 & 0 & 0 & \frac{1}{a} & 0\\ 0 & 0 & 0 & 1 & c & 0 \end{pmatrix} \middle| \begin{array}{l} a\in\mathbb{C}^*,\\ b,c\in\mathbb{C}\end{array}\right\}.$$

2. The set of anti–involutions of the pair 2.5.1 has the form:

$$\mathcal{I}=\left\{ I=\begin{pmatrix} \alpha & 0 & 0 & 0 & \alpha\gamma^2 & 2\alpha\gamma\\ 0 & \alpha & 0 & 2\alpha\beta & \alpha\beta^2 & 0\\ 0 & 0 & \alpha & -\alpha\gamma & -\alpha\beta\gamma & -\alpha\beta\\ 0 & 0 & 0 & 1 & \beta & 0\\ 0 & 0 & 0 & 0 & \frac{1}{\alpha} & 0\\ 0 & 0 & 0 & 0 & \gamma & 1 \end{pmatrix}, \text{ or } \begin{pmatrix} 0 & \alpha & 0 & 2\alpha\gamma & \alpha\gamma^2 & 0\\ \alpha & 0 & 0 & 0 & \alpha\beta^2 & 2\alpha\beta\\ 0 & 0 & \alpha & -\alpha\beta & -\alpha\beta\gamma & -\alpha\gamma\\ 0 & 0 & 0 & 0 & \beta & 1\\ 0 & 0 & 0 & 0 & \frac{1}{\alpha} & 0\\ 0 & 0 & 0 & 1 & \gamma & 0 \end{pmatrix} \middle| \begin{array}{l} I\overline{I}=E,\\ \alpha\in\mathbb{C}^*,\\ \beta,\gamma\in\mathbb{C}\end{array}\right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.5.1 is conjugate to one and only one of the following:

$$I_1=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a,b,x,y,z,t)$, $I_2(X) = (\overline{b},\overline{a},\overline{x},\overline{t},\overline{z},\overline{y})$, $I_2(X) = X$. It follows that $X = (a,\overline{a},x,y,z,\overline{y})$, where $a,y \in \mathbb{C}$, $x,z \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= e_1+e_2, & e_2' &= ie_1-ie_2,\\ u_1' &= u_1, & u_2' &= \frac{1}{2}u_2+\frac{1}{2}u_4,\\ u_3' &= \frac{1}{2}u_3, & u_4' &= \frac{i}{2}u_2-\frac{i}{2}u_4. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $2.5^2.1$.

Proposition 2.5.2. *Any real form of the pair* 2.5.2 *is equivalent to one and only one pair:*

$$2.5^1.2.$$

Proof. 1. The group of automorphisms of the pair 2.5.2 has the form:

$$\mathcal{A} = \left\{ \left(\begin{array}{cccccc} a & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 2db & db^2 & 0 \\ 0 & 0 & a & -dc & -dbc & -ab \\ 0 & 0 & 0 & 1 & b & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{d} & 0 \\ 0 & 0 & 0 & 0 & c & \frac{a}{d} \end{array} \right) \middle| \; a, d \in \mathbb{C}^*, \, b, c \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 2.5.2 has the form:

$$\mathcal{I} = \left\{ I = \left(\begin{array}{cccccc} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & 2\delta\beta & \delta\beta^2 & 0 \\ 0 & 0 & \alpha & -\delta\gamma & -\delta\beta\gamma & -\alpha\beta \\ 0 & 0 & 0 & 1 & \beta & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\delta} & 0 \\ 0 & 0 & 0 & 0 & \gamma & \frac{\alpha}{\delta} \end{array} \right) \middle| \; \begin{array}{c} I\overline{I} = E, \, \alpha, \delta \in \mathbb{C}^*, \\ \beta, \gamma \in \mathbb{C} \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.5.2 is conjugate to one and only one:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 2.5.3. *Any real form of the pair* 2.5.3 *is equivalent to one and only one of the following pairs:*

$$2.5^1.3, \quad 2.5^2.2 \ (s > 0).$$

Proof. 1. The group of automorphisms of the pair 2.5.3 has the form:
if $h \neq 0$ or $h = k = 0$

$$\mathcal{A} = \left\{ \left(\begin{array}{cccccc} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a^2 & -ac & d & -ab \\ 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & c & a \end{array} \right) \middle| \; a \in \mathbb{C}^*, \, b, c, d \in \mathbb{C} \right\},$$

if $h = 0,\ k \neq 0$

$$\mathcal{A} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a^2 & -ac & d & -ab \\ 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & c & a \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & a & 0 & 0 & 0 & 0 \\ -ka & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -ka^2 & -ab & d & kac \\ 0 & 0 & 0 & 0 & b & -ka \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a & c & 0 \end{pmatrix} \middle| \begin{array}{c} a \in \mathbb{C}^*, \\ b, c, d \in \mathbb{C} \end{array} \right\}.$$

2. The set of anti–involutions of the pair 2.5.3 has the form:
if $g, h, k \in \mathbb{R}$

$$\mathcal{I} = \left\{ I = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha^2 & -\alpha\gamma & \delta & -\alpha\beta \\ 0 & 0 & 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \gamma & \alpha \end{pmatrix} \middle| \begin{array}{c} I\overline{I} = E,\ \alpha \in \mathbb{C}^*, \\ \beta, \gamma, \delta \in \mathbb{C} \end{array} \right\},$$

if $h = \overline{g} - g,\ k \in \mathbb{R},\ k < 0$

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 \\ -k\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k\alpha^2 & -\alpha\beta & \delta & k\alpha\gamma \\ 0 & 0 & 0 & 0 & \beta & -k\alpha \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \alpha & \gamma & 0 \end{pmatrix} \middle| \begin{array}{c} I\overline{I} = E,\ \alpha \in \mathbb{C}^*, \\ \beta, \gamma, \delta \in \mathbb{C} \end{array} \right\}.$$

For other values of g, h, k there exist no anti–involution.

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.5.3 is conjugate to one and only one of the following:
if $g, h, k \in \mathbb{R}$

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

if $h = \overline{g} - g,\ k \in \mathbb{R},\ k < 0$

$$I_2 = \begin{pmatrix} 0 & \frac{1}{\sqrt{-k}} & 0 & 0 & 0 & 0 \\ \sqrt{-k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{-k} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{-k}} & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, b, x, y, z, t)$, $I_2(X) = (\frac{\bar{b}}{\sqrt{-k}}, \sqrt{-k}\bar{a}, \bar{x}, \sqrt{-k}\bar{t}, \bar{z}, \frac{\bar{y}}{\sqrt{-k}})$, $I_2(X) = X$. It follows that $X = (a, \sqrt{-k}\bar{a}, x, \sqrt{-k}\bar{t}, z, t)$, where $a, t \in \mathbb{C}$, $x, z \in \mathbb{R}$.

Let $g = p - ir$. Consider the following basis of the set of fixed points:

$$\begin{aligned}
e_1' &= e_1 + \sqrt{-k}e_2, \\
e_2' &= ie_1 - i\sqrt{-k}e_2, \\
u_1' &= 2\sqrt{-k}u_1, \\
u_2' &= \sqrt{-k}u_2 + u_4, \\
u_3' &= u_3, \\
u_4' &= i\sqrt{-k}u_2 - iu_4.
\end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $2.5^2.2$ $(s > 0)$, where $s = \sqrt{-k}$.

Proposition 2.5.4. *Any real form of the pair 2.5.4 is equivalent to one and only one of the following pairs:*

$$2.5^1.4, \quad 2.5^2.2\ (s = 0).$$

Proof. 1. The group of automorphisms of the pair 2.5.4 has the form:
if $h \neq 0$

$$\mathcal{A} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & ad & -dc & f & -ab \\ 0 & 0 & 0 & d & b & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & c & a \end{pmatrix} \middle| \; a, d \in \mathbb{C}^*,\ b, c, f \in \mathbb{C} \right\},$$

if $h = 0$

$$\mathcal{A} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & ad & -dc & f & -ab \\ 0 & 0 & 0 & d & b & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & c & a \end{pmatrix}, \begin{pmatrix} 0 & d & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ad & -db & f & -ac \\ 0 & 0 & 0 & 0 & b & a \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d & c & 0 \end{pmatrix} \middle| \; \begin{array}{c} a, d \in \mathbb{C}^*, \\ b, c, f \in \mathbb{C} \end{array} \right\}.$$

2. The set of anti–involutions of the pair 2.5.4 has the form:
if $g, h \in \mathbb{R}$

$$\mathcal{I} = \left\{ I = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha\delta & -\delta\gamma & \varepsilon & -\alpha\beta \\ 0 & 0 & 0 & \delta & \beta & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \gamma & \alpha \end{pmatrix} \middle| \; \begin{array}{c} I\bar{I} = E,\ \alpha, \delta \in \mathbb{C}^*, \\ \beta, \gamma, \varepsilon \in \mathbb{C} \end{array} \right\},$$

if $h = \overline{g} - g$

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 0 & \delta & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha\delta & -\delta\beta & \varepsilon & -\alpha\gamma \\ 0 & 0 & 0 & 0 & \beta & \alpha \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \delta & \gamma & 0 \end{pmatrix} \;\middle|\; \begin{matrix} I\overline{I} = E,\ \alpha, \delta \in \mathbb{C}^*, \\ \beta, \gamma, \varepsilon \in \mathbb{C} \end{matrix} \right\}.$$

For other values of g, h there exist no anti–involutions.

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.5.4 is conjugate to one and only one of the following:

If $g, h \in \mathbb{R}$

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

If $h = \overline{g} - g$

$$I_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, b, x, y, z, t)$, $I_2(X) = (\overline{b}, \overline{a}, \overline{x}, \overline{t}, \overline{z}, \overline{y})$, $I_2(X) = X$. It follows that $X = (a, \overline{a}, x, y, z, \overline{y})$, where $a, y \in \mathbb{C}$, $x, z \in \mathbb{R}$.

Let $g = p - ir$. Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= e_1 + e_2, & e_2' &= ie_1 - ie_2, \\ u_1' &= 2u_1, & u_2' &= u_2 + u_4, \\ u_3' &= u_3, & u_4' &= iu_2 - iu_4. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $2.5^2.2$ ($s = 0$).

Proposition 2.5.5. *Any real form of the pair* 2.5.5 *is equivalent to one and only one of the following pairs:*

$$2.5^1.5, \quad 2.5^2.3 \ (s > 0).$$

Proof. 1. The group of automorphisms of the pair 2.5.5 has the form:
if $k = 0$

$$\mathcal{A} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a^2 & -ac & d & -ab \\ 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & c & a \end{pmatrix} \middle| \; a \in \mathbb{C}^*,\, b, c, d \in \mathbb{C} \right\},$$

if $k \neq 0$

$$\mathcal{A} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a^2 & -ac & d & -ab \\ 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & c & a \end{pmatrix}, \begin{pmatrix} 0 & a & 0 & 0 & 0 & 0 \\ -ka & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ka^2 & -ab & d & kac \\ 0 & 0 & 0 & 0 & b & ka \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -a & c & 0 \end{pmatrix} \middle| \; \begin{matrix} a \in \mathbb{C}^*, \\ b, c, d \in \mathbb{C} \end{matrix} \right\}.$$

2. The set of anti–involutions of the pair 2.5.5 has the form:
if $g, k \in \mathbb{R}$, $k \geqslant 0$

$$\mathcal{I} = \left\{ I = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha^2 & -\alpha\gamma & \delta & -\alpha\beta \\ 0 & 0 & 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \gamma & \alpha \end{pmatrix} \middle| \; \begin{matrix} I\bar{I} = E,\, \alpha \in \mathbb{C}^*, \\ \beta, \gamma, \delta \in \mathbb{C} \end{matrix} \right\}.$$

if $g, k \in \mathbb{R}$, $k < 0$

$$\mathcal{I} = \left\{ I = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha^2 & -\alpha\gamma & \delta & -\alpha\beta \\ 0 & 0 & 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \gamma & \alpha \end{pmatrix}, \text{ or } \begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 \\ -k\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k\alpha^2 & -\alpha\beta & \delta & k\alpha\gamma \\ 0 & 0 & 0 & 0 & \beta & k\alpha \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\alpha & \gamma & 0 \end{pmatrix} \middle| \; I\bar{I} = E,\, \alpha \in \mathbb{C}^*, \beta, \gamma, \delta \in \mathbb{C} \right\}.$$

For other values of g, k there exist no anti–involution.

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\bar{A}^{-1}$ we obtain that any anti–involution of the pair 2.5.5 is conjugate to one and only one of the following:

if $g, k \in \mathbb{R},\ k \geqslant 0$

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

if $g, k \in \mathbb{R},\ k < 0$

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & \frac{1}{\sqrt{-k}} & 0 & 0 & 0 & 0 \\ \sqrt{-k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{-k} \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{-k}} & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, b, x, y, z, t)$, $I_2(X) = (\frac{\bar{b}}{\sqrt{-k}}, \sqrt{-k}\bar{a}, -\bar{x}, -\sqrt{-k}\bar{t}, -\bar{z}, -\frac{\bar{y}}{\sqrt{-k}})$, $I_2(X) = X$. It follows that $X = (a, \sqrt{-k}\bar{a}, x, -\sqrt{-k}\bar{t}, z, t)$, where $a, t \in \mathbb{C}$, $x, z \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= e_1 + \sqrt{-k}e_2, \\ e_2' &= ie_1 - i\sqrt{-k}e_2, \\ u_1' &= 2i\sqrt{-k}u_1, \\ u_2' &= i\sqrt{-k}u_2 + iu_4, \\ u_3' &= iu_3, \\ u_4' &= u_4 - \sqrt{-k}u_2. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $2.5^2.3$ $(s > 0)$, where $s = \sqrt{-k},\ r = g$.

Proposition 2.5.6. *Any real form of the pair 2.5.6 is equivalent to one and only one of the following pairs:*

$$2.5^1.6, \quad 2.5^2.3\ (s = 0).$$

Proof. 1. The group of automorphisms of the pair 2.5.6 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & ad & -dc & f & -ab \\ 0 & 0 & 0 & d & b & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & c & a \end{pmatrix}, \begin{pmatrix} 0 & d & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -ad & -db & f & -ac \\ 0 & 0 & 0 & 0 & b & -a \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -d & c & 0 \end{pmatrix} \middle| \begin{array}{l} a, d \in \mathbb{C}^*, \\ b, c, f \in \mathbb{C} \end{array} \right\}.$$

2. The set of anti–involutions of the pair 2.5.6 has the form:
if $g \in \mathbb{R}$

$$\mathcal{I} = \left\{ I = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha\delta & -\delta\gamma & \varepsilon & -\alpha\beta \\ 0 & 0 & 0 & \delta & \beta & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \gamma & \alpha \end{pmatrix}, \text{ or} \right.$$

$$\left. \begin{pmatrix} 0 & \delta & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha\delta & -\delta\beta & \varepsilon & -\alpha\gamma \\ 0 & 0 & 0 & 0 & \beta & -\alpha \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\delta & \gamma & 0 \end{pmatrix} \middle| \begin{array}{c} I\overline{I} = E, \\ \alpha, \delta \in \mathbb{C}^*, \\ \beta, \gamma, \varepsilon \in \mathbb{C} \end{array} \right\}.$$

For other values of g there exist no anti–involution.

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.5.6 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, b, x, y, z, t)$, $I_2(X) = (\overline{b}, \overline{a}, -\overline{x}, -\overline{t}, -\overline{z}, -\overline{y})$, $I_2(X) = X$. It follows that $X = (a, \overline{a}, x, y, z, -\overline{y})$, where $a, y \in \mathbb{C}$, $x, z \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= e_1 + e_2, & e_2' &= ie_1 - ie_2, \\ u_1' &= 2iu_1, & u_2' &= iu_2 + iu_4, \\ u_3' &= iu_3, & u_4' &= u_4 - u_2. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $2.5^2.3$ $(s = 0)$, where $r = g$.

Proposition 2.5.7. *Any real form of the pair* 2.5.7 *is equivalent to one and only one of the following pairs:*

$$2.5^1.7, \quad 2.5^1.8, \quad 2.5^2.4\ (s > 0), \quad 2.5^2.5\ (s > 0).$$

Proof. 1. The group of automorphisms of the pair 2.5.7 has the form:
if $k = 0$

$$\mathcal{A} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a^2 & -ac & d & -ab \\ 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & c & a \end{pmatrix}, \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & -a^2 & -ac & d & -ab \\ 0 & 0 & 0 & -a & b & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & c & -a \end{pmatrix} \middle| \begin{array}{c} a \in \mathbb{C}^*, \\ b, c, d \in \mathbb{C} \end{array} \right\},$$

if $k \neq 0$

$$\mathcal{A}=\left\{\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0\\ 0 & a & 0 & 0 & 0 & 0\\ 0 & 0 & a^2 & -ac & d & -ab\\ 0 & 0 & 0 & a & b & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & c & a \end{pmatrix},\right.$$

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0\\ 0 & a & 0 & 0 & 0 & 0\\ 0 & 0 & -a^2 & -ac & d & -ab\\ 0 & 0 & 0 & -a & b & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & c & -a \end{pmatrix}, \begin{pmatrix} 0 & a & 0 & 0 & 0 & 0\\ -ka & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -ka^2 & -ab & d & kac\\ 0 & 0 & 0 & 0 & b & -ka\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & a & c & 0 \end{pmatrix},$$

$$\left.\begin{pmatrix} 0 & a & 0 & 0 & 0 & 0\\ -ka & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & ka^2 & -ab & d & kac\\ 0 & 0 & 0 & 0 & b & ka\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -a & c & 0 \end{pmatrix} \;\middle|\; \begin{array}{c} a \in \mathbb{C}^*,\\ b, c, d \in \mathbb{C} \end{array}\right\}.$$

2. The set of anti–involutions of the pair 2.5.7 has the form:
if $k \in \mathbb{R},\ k \geqslant 0$

$$\mathcal{I}=\left\{ I=\begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0\\ 0 & \alpha & 0 & 0 & 0 & 0\\ 0 & 0 & \alpha^2 & -\alpha\gamma & \delta & -\alpha\beta\\ 0 & 0 & 0 & \alpha & \beta & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & \gamma & \alpha \end{pmatrix}, \text{ or}\right.$$

$$\left.\begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0\\ 0 & \alpha & 0 & 0 & 0 & 0\\ 0 & 0 & -\alpha^2 & -\alpha\gamma & \delta & -\alpha\beta\\ 0 & 0 & 0 & -\alpha & \beta & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & \gamma & -\alpha \end{pmatrix} \;\middle|\; \begin{array}{c} I\bar{I}=E,\ \alpha \in \mathbb{C}^*,\\ \beta, \gamma, \delta \in \mathbb{C} \end{array}\right\},$$

if $k \in \mathbb{R},\ k < 0$

$$\mathcal{I}=\left\{ I \in \left\{ \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0\\ 0 & \alpha & 0 & 0 & 0 & 0\\ 0 & 0 & \alpha^2 & -\alpha\gamma & \delta & -\alpha\beta\\ 0 & 0 & 0 & \alpha & \beta & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & \gamma & \alpha \end{pmatrix},\right.\right.$$

$$\begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha^2 & -\alpha\gamma & \delta & -\alpha\beta \\ 0 & 0 & 0 & -\alpha & \beta & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \gamma & -\alpha \end{pmatrix}, \begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 \\ -k\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k\alpha^2 & -\alpha\beta & \delta & k\alpha\gamma \\ 0 & 0 & 0 & 0 & \beta & -k\alpha \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \alpha & \gamma & 0 \end{pmatrix},$$

$$\left.\begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 \\ -k\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k\alpha^2 & -\alpha\beta & \delta & k\alpha\gamma \\ 0 & 0 & 0 & 0 & \beta & k\alpha \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\alpha & \gamma & 0 \end{pmatrix}\right\} \left| \begin{array}{c} I\overline{I} = E,\ \alpha \in \mathbb{C}^*, \\ \beta, \gamma, \delta \in \mathbb{C} \end{array} \right\}.$$

For other values of k there exist no anti–involutions.

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.5.7 is conjugate to one and only one of the following:

if $k \in \mathbb{R}$, $k \geqslant 0$

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

if $k \in \mathbb{R}$, $k < 0$

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$I_3 = \begin{pmatrix} 0 & \frac{1}{\sqrt{-k}} & 0 & 0 & 0 & 0 \\ \sqrt{-k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{-k} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{-k}} & 0 & 0 \end{pmatrix},$$

$$I_4 = \begin{pmatrix} 0 & \frac{1}{\sqrt{-k}} & 0 & 0 & 0 & 0 \\ \sqrt{-k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{-k} \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{-k}} & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, b, x, y, z, t)$, $I_2(X) = (\overline{a}, \overline{b}, -\overline{x}, -\overline{y}, -\overline{z}, -\overline{t})$, $I_2(X) = X$. It follows that $X = (a, b, x, y, z, t)$, where $a, b \in \mathbb{R}$, $x, y, z, t \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= -e_1, & e_2' &= e_2, \\ u_1' &= -iu_1, & u_2' &= iu_2, \\ u_3' &= iu_3, & u_4' &= -iu_4. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $2.5^1.8$.

5. Find the set of fixed points of the mapping I_3.

Let $X = (a, b, x, y, z, t)$, $I_3(X) = (\frac{\overline{b}}{\sqrt{-k}}, \sqrt{-k}\overline{a}, \overline{x}, \sqrt{-k}\overline{t}, \overline{z}, \frac{\overline{y}}{\sqrt{-k}})$, $I_3(X) = X$. It follows that $X = (a, \sqrt{-k}\overline{a}, x, \sqrt{-k}\overline{t}, z, t)$, where $a, t \in \mathbb{C}$, $x, z \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= e_1 + \sqrt{-k}e_2, \\ e_2' &= ie_1 - i\sqrt{-k}e_2, \\ u_1' &= 2\sqrt{-k}u_1, \\ u_2' &= \sqrt{-k}u_2 + u_4, \\ u_3' &= u_3, \\ u_4' &= i\sqrt{-k}u_2 - iu_4. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_3 has the form $2.5^2.4$ $(s > 0)$, where $s = \sqrt{-k}$.

6. Find the set of fixed points of the mapping I_4.

Let $X = (a, b, x, y, z, t)$, $I_4(X) = (\frac{\overline{b}}{\sqrt{-k}}, \sqrt{-k}\overline{a}, -\overline{x}, -\sqrt{-k}\overline{t}, -\overline{z}, -\frac{\overline{y}}{\sqrt{-k}})$, $I_4(X) = X$. It follows that $X = (a, \sqrt{-k}\overline{a}, x, -\sqrt{-k}\overline{t}, z, t)$, where $a, t \in \mathbb{C}$, $x, z \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= e_1 + \sqrt{-k}e_2, \\ e_2' &= ie_1 - i\sqrt{-k}e_2, \\ u_1' &= 2i\sqrt{-k}u_1, \\ u_2' &= i\sqrt{-k}u_2 + iu_4, \\ u_3' &= iu_3, \\ u_4' &= u_4 - \sqrt{-k}u_2. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_4 has the form $2.5^2.5$ $(s > 0)$, where $s = \sqrt{-k}$.

Proposition 2.5.8. *Any real form of the pair* 2.5.8 *is equivalent to one and only one of the following pairs:*

$$2.5^1.9, \quad 2.5^1.10, \quad 2.5^2.4\ (s=0), \quad 2.5^2.5\ (s=0).$$

Proof. 1. The group of automorphisms of the pair 2.5.8 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & ad & -dc & f & -ab \\ 0 & 0 & 0 & d & b & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & c & a \end{pmatrix}, \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & -ad & -dc & f & -ab \\ 0 & 0 & 0 & -d & b & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & c & -a \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & d & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ad & -db & f & -ac \\ 0 & 0 & 0 & 0 & b & a \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d & c & 0 \end{pmatrix}, \begin{pmatrix} 0 & d & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -ad & -db & f & -ac \\ 0 & 0 & 0 & 0 & b & -a \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -d & c & 0 \end{pmatrix} \;\middle|\; \begin{matrix} a, d \in \mathbb{C}^*, \\ b, c, f \in \mathbb{C} \end{matrix} \right\}.$$

2. The set of anti–involutions of the pair 2.5.8 has the form:

$$\mathcal{I} = \left\{ I \in \left\{ \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha\delta & -\delta\gamma & \varepsilon & -\alpha\beta \\ 0 & 0 & 0 & \delta & \beta & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \gamma & \alpha \end{pmatrix}, \right. \right.$$
$$\begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha\delta & -\delta\gamma & \varepsilon & -\alpha\beta \\ 0 & 0 & 0 & -\delta & \beta & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \gamma & -\alpha \end{pmatrix}, \begin{pmatrix} 0 & \delta & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha\delta & -\delta\beta & \varepsilon & -\alpha\gamma \\ 0 & 0 & 0 & 0 & \beta & \alpha \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \delta & \gamma & 0 \end{pmatrix},$$
$$\left. \left. \begin{pmatrix} 0 & \delta & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha\delta & -\delta\beta & \varepsilon & -\alpha\gamma \\ 0 & 0 & 0 & 0 & \beta & -\alpha \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\delta & \gamma & 0 \end{pmatrix} \right\} \;\middle|\; \begin{matrix} I\overline{I} = E, \\ \alpha, \delta \in \mathbb{C}^*, \\ \beta, \gamma, \varepsilon \in \mathbb{C} \end{matrix} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.5.8 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$I_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad I_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, b, x, y, z, t)$, $I_2(X) = (\overline{a}, \overline{b}, -\overline{x}, -\overline{y}, -\overline{z}, -\overline{t})$, $I_2(X) = X$. It follows that $X = (a, b, x, y, z, t)$, where $a, b \in \mathbb{R}$, $x, y, z, t \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= -e_1, & e_2' &= e_2, \\ u_1' &= -iu_1, & u_2' &= iu_2, \\ u_3' &= iu_3, & u_4' &= -iu_4. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $2.5^1.10$.

5. Find the set of fixed points of the mapping I_3.

Let $X = (a, b, x, y, z, t)$, $I_3(X) = (\overline{b}, \overline{a}, \overline{x}, \overline{t}, \overline{z}, \overline{y})$, $I_3(X) = X$. It follows that $X = (a, \overline{a}, x, y, z, \overline{y})$, where $a, y \in \mathbb{C}$, $x, z \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$e_1' = e_1 + e_2, \quad e_2' = ie_1 - ie_2,$$

$$\begin{aligned} u_1' &= 2u_1, & u_2' &= u_2 + u_4, \\ u_3' &= u_3, & u_4' &= iu_2 - iu_4. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_3 has the form $2.5^2.4$ $(s = 0)$.

6. Find the set of fixed points of the mapping I_4.

Let $X = (a, b, x, y, z, t)$, $I_4(X) = (\overline{b}, \overline{a}, -\overline{x}, -\overline{t}, -\overline{z}, -\overline{y})$, $I_4(X) = X$. It follows that $X = (a, \overline{a}, x, y, z, -\overline{y})$, where $a, y \in \mathbb{C}$, $x, z \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e_1' = e_1 + e_2, \quad e_2' = ie_1 - ie_2,$$

$$\begin{aligned} u_1' &= 2iu_1, & u_2' &= iu_2 + iu_4, \\ u_3' &= iu_3, & u_4' &= u_4 - u_2. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_4 has the form $2.5^2.5$ $(s = 0)$.

Proposition 2.5.9. *Any real form of the pair* 2.5.9 *is equivalent to one and only one of the following pairs:*

$$2.5^1.11, \quad 2.5^1.12, \quad 2.5^2.6.$$

Proof. 1. The group of automorphisms of the pair 2.5.9 has the form:

$$\mathcal{A}=\left\{\begin{pmatrix} a&0&0&0&0&0\\ 0&a&0&0&0&0\\ 0&0&a^2&-ac&d&-ab\\ 0&0&0&a&b&0\\ 0&0&0&0&1&0\\ 0&0&0&0&c&a \end{pmatrix},\begin{pmatrix} a&0&0&0&0&0\\ 0&a&0&0&0&0\\ 0&0&-a^2&-ac&d&-ab\\ 0&0&0&-a&b&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&c&-a \end{pmatrix},\right.$$
$$\begin{pmatrix} -a&0&0&0&0&0\\ 0&a&0&0&0&0\\ 0&0&-ia^2&-ac&d&ab\\ 0&0&0&ia&b&0\\ 0&0&0&0&i&0\\ 0&0&0&0&c&-ia \end{pmatrix},\begin{pmatrix} -a&0&0&0&0&0\\ 0&a&0&0&0&0\\ 0&0&ia^2&-ac&d&ab\\ 0&0&0&-ia&b&0\\ 0&0&0&0&-i&0\\ 0&0&0&0&c&ia \end{pmatrix},$$
$$\begin{pmatrix} 0&a&0&0&0&0\\ -a&0&0&0&0&0\\ 0&0&-a^2&-ab&d&ac\\ 0&0&0&0&b&-a\\ 0&0&0&0&1&0\\ 0&0&0&a&c&0 \end{pmatrix},\begin{pmatrix} 0&a&0&0&0&0\\ -a&0&0&0&0&0\\ 0&0&a^2&-ab&d&ac\\ 0&0&0&0&b&a\\ 0&0&0&0&-1&0\\ 0&0&0&-a&c&0 \end{pmatrix},$$
$$\left.\begin{pmatrix} 0&a&0&0&0&0\\ a&0&0&0&0&0\\ 0&0&ia^2&-ab&d&-ac\\ 0&0&0&0&b&ia\\ 0&0&0&0&i&0\\ 0&0&0&ia&c&0 \end{pmatrix},\begin{pmatrix} 0&a&0&0&0&0\\ a&0&0&0&0&0\\ 0&0&-ia^2&-ab&d&-ac\\ 0&0&0&0&b&-ia\\ 0&0&0&0&-i&0\\ 0&0&0&-ia&c&0 \end{pmatrix}\;\middle|\; \begin{matrix} a\in\mathbb{C}^*,\\ b,c,d\in\mathbb{C}\end{matrix}\right\}.$$

2. The set of anti–involutions of the pair 2.5.9 has the form:

$$\mathcal{I}=\left\{I\in\left\{\begin{pmatrix} \alpha&0&0&0&0&0\\ 0&\alpha&0&0&0&0\\ 0&0&\alpha^2&-\alpha\gamma&\delta&-\alpha\beta\\ 0&0&0&\alpha&\beta&0\\ 0&0&0&0&1&0\\ 0&0&0&0&\gamma&\alpha \end{pmatrix},\right.\right.$$
$$\begin{pmatrix} \alpha&0&0&0&0&0\\ 0&\alpha&0&0&0&0\\ 0&0&-\alpha^2&-\alpha\gamma&\delta&-\alpha\beta\\ 0&0&0&-\alpha&\beta&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&\gamma&-\alpha \end{pmatrix},\begin{pmatrix} -\alpha&0&0&0&0&0\\ 0&\alpha&0&0&0&0\\ 0&0&-i\alpha^2&-\alpha\gamma&\delta&\alpha\beta\\ 0&0&0&i\alpha&\beta&0\\ 0&0&0&0&i&0\\ 0&0&0&0&\gamma&-i\alpha \end{pmatrix},$$
$$\begin{pmatrix} -\alpha&0&0&0&0&0\\ 0&\alpha&0&0&0&0\\ 0&0&i\alpha^2&-\alpha\gamma&\delta&\alpha\beta\\ 0&0&0&-i\alpha&\beta&0\\ 0&0&0&0&-i&0\\ 0&0&0&0&\gamma&i\alpha \end{pmatrix},\begin{pmatrix} 0&\alpha&0&0&0&0\\ \alpha&0&0&0&0&0\\ 0&0&i\alpha^2&-\alpha\beta&\delta&-\alpha\gamma\\ 0&0&0&0&\beta&i\alpha\\ 0&0&0&0&i&0\\ 0&0&0&i\alpha&\gamma&0 \end{pmatrix},$$

$$\left\{\left.\begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\alpha^2 & -\alpha\beta & \delta & -\alpha\gamma \\ 0 & 0 & 0 & 0 & \beta & -i\alpha \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i\alpha & \gamma & 0 \end{pmatrix}\right| \begin{array}{c} I\overline{I} = E, \\ \alpha \in \mathbb{C}^*, \\ \beta, \gamma, \delta \in \mathbb{C} \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.5.9 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \end{pmatrix},$$

$$I_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & i & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, b, x, y, z, t)$, $I_2(X) = (-\overline{a}, \overline{b}, -i\overline{x}, i\overline{y}, i\overline{z}, -i\overline{t})$, $I_2(X) = X$. It follows that $X = (a, b, x - ix, y + iy, z + iz, t - it)$, where $a \in \mathbb{R}i$, $b, x, y, z, t \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$\begin{aligned} e_1' &= -ie_1, & e_2' &= e_2, \\ u_1' &= \frac{(1-i)}{\sqrt{2}}u_1, & u_2' &= \frac{1+i}{\sqrt{2}}u_2, \\ u_3' &= -\frac{1+i}{\sqrt{2}}u_3, & u_4' &= \frac{1-i}{\sqrt{2}}u_4. \end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $2.5^1.12$.

5. Find the set of fixed points of the mapping I_3.

Let $X = (a, b, x, y, z, t)$, $I_3(X) = (\overline{b}, \overline{a}, i\overline{x}, i\overline{t}, i\overline{z}, i\overline{y})$, $I_3(X) = X$. It follows that $X = (a, \overline{a}, x + ix, y, z + iz, i\overline{y})$, where $a, y \in \mathbb{C}$, $x, z \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$\begin{aligned}
e_1' &= e_1 + e_2,\\
e_2' &= ie_1 - ie_2,\\
u_1' &= \frac{2(1+i)}{\sqrt{2}}u_1,\\
u_2' &= \frac{1+i}{\sqrt{2}}u_2 + \frac{1+i}{\sqrt{2}}u_4,\\
u_3' &= \frac{1+i}{\sqrt{2}}u_3,\\
u_4' &= \frac{i-1}{\sqrt{2}}u_2 + \frac{1-i}{\sqrt{2}}u_4.
\end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_3 has the form $2.5^2.6$.

Proposition 2.5.10. *Any real form of the pair* 2.5.10 *is equivalent to one and only one pair:*

$$2.5^1.13.$$

Proof. 1. The group of automorphisms of the pair 2.5.10 has the form:

$$\mathcal{A} = \left\{ \left. \begin{pmatrix} ad^2 & 0 & 0 & 0 & 0 & 0\\ 0 & a & 0 & 0 & 0 & 0\\ 0 & 0 & a^2d^3 & -ac & f & -ad^2b\\ 0 & 0 & 0 & ad & b & 0\\ 0 & 0 & 0 & 0 & d & 0\\ 0 & 0 & 0 & 0 & c & ad^3 \end{pmatrix} \right| a, d \in \mathbb{C}^*,\ b, c, f \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 2.5.10 has the form:

$$\mathcal{I} = \left\{ I = \left. \begin{pmatrix} \alpha\delta^2 & 0 & 0 & 0 & 0 & 0\\ 0 & \alpha & 0 & 0 & 0 & 0\\ 0 & 0 & \alpha^2\delta^3 & -\alpha\gamma & \varepsilon & -\alpha\delta^2\beta\\ 0 & 0 & 0 & \alpha\delta & \beta & 0\\ 0 & 0 & 0 & 0 & \delta & 0\\ 0 & 0 & 0 & 0 & \gamma & \alpha\delta^3 \end{pmatrix} \right| \begin{array}{c} I\overline{I} = E,\ \alpha, \delta \in \mathbb{C}^*,\\ \beta, \gamma, \varepsilon \in \mathbb{C} \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.5.10 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 2.5.11. *Any real form of the pair* 2.5.11 *is equivalent to one and only one of the following pairs:*

$$2.5^1.14, \quad 2.5^2.7.$$

Proof. 1. The group of automorphisms of the pair 2.5.11 has the form:

$$\mathcal{A}=\left\{\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & adf & -dc & l & -ab \\ 0 & 0 & 0 & df & b & 0 \\ 0 & 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & c & af \end{pmatrix}, \begin{pmatrix} 0 & d & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & adf & -db & l & -ac \\ 0 & 0 & 0 & 0 & b & af \\ 0 & 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & df & c & 0 \end{pmatrix} \middle| \begin{array}{c} a, d, f \in \mathbb{C}^*, \\ b, c, l \in \mathbb{C} \end{array} \right\}.$$

2. The set of anti–involutions of the pair 2.5.11 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha\delta\varepsilon & -\delta\gamma & \eta & -\alpha\beta \\ 0 & 0 & 0 & \delta\varepsilon & \beta & 0 \\ 0 & 0 & 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 & \gamma & \alpha\varepsilon \end{pmatrix}, \text{ or } \begin{pmatrix} 0 & \delta & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha\delta\varepsilon & -\delta\beta & \eta & -\alpha\gamma \\ 0 & 0 & 0 & 0 & \beta & \alpha\varepsilon \\ 0 & 0 & 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & \delta\varepsilon & \gamma & 0 \end{pmatrix} \middle| \begin{array}{c} I\overline{I} = E,\ \alpha, \delta, \varepsilon \in \mathbb{C}^*, \\ \beta, \gamma, \eta \in \mathbb{C} \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 2.5.11 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, b, x, y, z, t)$, $I_2(X) = (\overline{b}, \overline{a}, \overline{x}, \overline{t}, \overline{z}, \overline{y})$, $I_2(X) = X$. It follows that $X = (a, \overline{a}, x, y, z, \overline{y})$, where $a, y \in \mathbb{C}$, $x, z \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$e_1' = e_1 + e_2, \quad e_2' = ie_1 - ie_2,$$

$$u_1' = 2u_1, \quad u_2' = u_2 + u_4,$$
$$u_3' = u_3, \quad u_4' = iu_2 - iu_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $2.5^2.7$.

3. Real forms of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ with solvable subalgebra $\mathfrak{g}$ Three- and four-dimensional cases

Theorem 3.1. *Any real form of the linear Lie algebra* 3.1 *is conjugate to one and only one of the following linear Lie algebras:*

$$3.1^1 \quad \begin{pmatrix} x & z & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -z & -y \end{pmatrix} \qquad 3.1^2 \quad \begin{pmatrix} x & -y & z & 0 \\ y & x & 0 & z \\ 0 & 0 & -x & -y \\ 0 & 0 & y & -x \end{pmatrix}$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 3.1^1 *is equivalent to one and only one pair:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	0	e_3	u_1	0	$-u_3$	0
e_2	0	0	$-e_3$	0	u_2	0	$-u_4$
e_3	$-e_3$	e_3	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0	0
u_2	0	$-u_2$	$-u_1$	0	0	0	0
u_3	u_3	0	u_4	0	0	0	0
u_4	0	u_4	0	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 3.1^2 *is equivalent to one and only one pair:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	u_2	$-u_1$	u_4	$-u_3$
e_3	$-2e_3$	0	0	0	0	u_1	u_2
u_1	$-u_1$	$-u_2$	0	0	0	0	0
u_2	$-u_2$	u_1	0	0	0	0	0
u_3	u_3	$-u_4$	$-u_1$	0	0	0	0
u_4	u_4	u_3	$-u_2$	0	0	0	0

Proof. 1. The group of automorphisms of the pair 3.1.1 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ a & -a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & bc & -ac & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & ad & bd \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & -a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -ad & -bd \\ 0 & 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & -bc & ac & 0 & 0 \end{pmatrix} \;\middle|\; b, c, d \in \mathbb{C}^*, a \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 3.1.1 has the form:

$$\mathcal{I}=\left\{ I=\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\ \alpha&-\alpha&\beta&0&0&0&0\\0&0&0&\beta\gamma&-\alpha\gamma&0&0\\0&0&0&0&\gamma&0&0\\0&0&0&0&0&\delta&0\\0&0&0&0&0&\alpha\delta&\beta\delta\end{pmatrix},\ \text{or}\right.$$

$$\left.\begin{pmatrix}0&-1&0&0&0&0&0\\-1&0&0&0&0&0&0\\ \alpha&-\alpha&\beta&0&0&0&0\\0&0&0&0&0&-\alpha\delta&-\beta\delta\\0&0&0&0&0&\delta&0\\0&0&0&0&\gamma&0&0\\0&0&0&-\beta\gamma&\alpha\gamma&0&0\end{pmatrix}\ \middle|\ I\overline{I}=E,\beta,\gamma,\delta\in\mathbb{C}^*,\alpha\in\mathbb{C}\right\}.$$

3. Let $A\in\mathcal{A}$, $I\in\mathcal{I}$. Using the operation $I\mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 3.1.1 is conjugate to one and only one of the following:

$$I_1=\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix},\qquad I_2=\begin{pmatrix}0&-1&0&0&0&0&0\\-1&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&0&0&-1\\0&0&0&0&0&1&0\\0&0&0&0&1&0&0\\0&0&0&-1&0&0&0\end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X=(a,b,c,x,y,z,t)$, $I_2(X)=(-\overline{b},-\overline{a},\overline{c},-\overline{t},\overline{z},\overline{y},-\overline{x})$, $I_2(X)=X$. It follows that $X=(a,-\overline{a},c,x,y,\overline{y},-\overline{x})$, where $a,x,y\in\mathbb{C}$, $c\in\mathbb{R}$.

Consider the following basis of the set of fixed points:

$$e_1'=e_1-e_2,\quad e_2'=ie_1+ie_2,\quad e_3'=e_3,$$

$$u_1'=u_1-u_4,\quad u_2'=iu_1+iu_4,$$
$$u_3'=u_2+u_3,\quad u_4'=iu_2-iu_3.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $3.1^2.1$.

Theorem 3.2. *Any real form of the linear Lie algebra 3.2 is conjugate to one and only one of the following linear Lie algebras:*

$$3.2^1\quad\begin{pmatrix}x&y&0&z\\0&\lambda x&-z&0\\0&0&-x&0\\0&0&-y&-\lambda x\end{pmatrix},\ \lambda\geqslant 0\qquad 3.2^2\quad\begin{pmatrix}x&y&0&-z\\0&0&-y&-\lambda x\\0&0&-x&0\\0&\lambda x&z&0\end{pmatrix},\ \lambda\geqslant 0$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 3.2^1 *is equivalent to one and only one of the following pairs:*

$\lambda = 0$

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	e_2	e_3	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	0	u_1	$-u_4$	$-2e_2$
e_3	$-e_3$	0	0	0	$-2e_3$	$-u_2$	u_1
u_1	$-u_1$	0	0	0	$2e_3-u_1$	u_2+u_4	$2e_2-u_1$
u_2	0	$-u_1$	$2e_3$	u_1-2e_3	0	$-2u_3$	u_2-u_4
u_3	u_3	u_4	u_2	$-u_2-u_4$	$2u_3$	0	$2u_3$
u_4	0	$2e_2$	$-u_1$	u_1-2e_2	u_4-u_2	$-2u_3$	0

2.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	e_2	e_3	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	0	u_1	$-u_4$	$-2e_2$
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	u_2	$-u_1$
u_2	0	$-u_1$	0	0	0	0	u_2
u_3	u_3	u_4	u_2	$-u_2$	0	0	$2u_3$
u_4	0	$2e_2$	$-u_1$	u_1	$-u_2$	$-2u_3$	0

$\lambda = 1$

3.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	0	u_1	$-u_4$	0
e_3	$-2e_3$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	e_2	0
u_3	u_3	u_4	u_2	0	$-e_2$	0	0
u_4	u_4	0	$-u_1$	0	0	0	0

$\lambda \geqslant 0$

4.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$(1-\lambda)e_2$	$(1+\lambda)e_3$	u_1	λu_2	$-u_3$	$-\lambda u_4$
e_2	$(\lambda-1)e_2$	0	0	0	u_1	$-u_4$	0
e_3	$-(1+\lambda)e_3$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	0	0
u_2	$-\lambda u_2$	$-u_1$	0	0	0	0	0
u_3	u_3	u_4	u_2	0	0	0	0
u_4	λu_4	0	$-u_1$	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 3.2^2 *is equivalent to one and only one of the following pairs:*

$\lambda = 0$

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	e_2	e_3	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	0	$-e_2+u_1$	$-u_2$	e_3
e_3	$-e_3$	0	0	0	$-e_3$	u_4	$-e_2-u_1$
u_1	$-u_1$	0	0	0	e_2-u_1	u_2	$-e_3$
u_2	0	e_2-u_1	e_3	$-e_2+u_1$	0	$-2u_3$	$-u_4$
u_3	u_3	u_2	$-u_4$	$-u_2$	$2u_3$	0	0
u_4	0	$-e_3$	e_2+u_1	e_3	u_4	0	0

$\lambda \geqslant 0$

2.

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$e_2-\lambda e_3$	$e_3+\lambda e_2$	u_1	λu_4	$-u_3$	$-\lambda u_2$
e_2	λe_3-e_2	0	0	0	u_1	$-u_2$	0
e_3	$-e_3-\lambda e_2$	0	0	0	0	u_4	$-u_1$
u_1	$-u_1$	0	0	0	0	0	0
u_2	$-\lambda u_4$	$-u_1$	0	0	0	0	0
u_3	u_3	u_2	$-u_4$	0	0	0	0
u_4	λu_2	0	u_1	0	0	0	0

The proof of the Theorem follows from Propositions 3.2.1–3.2.4.

Proposition 3.2.1. *Any real form of the pair* 3.2.1 *is equivalent to one and only one of the following pairs:*

$$3.2^1.1, \quad 3.2^2.1.$$

Proof. 1. The group of automorphisms of the pair 3.2.1 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & \frac{1}{a} & 0 & 0 & 0 & ab^2 & 2b \\ c & 0 & \frac{1}{a} & 0 & 2c & ac^2 & 0 \\ 0 & 0 & 0 & \frac{1}{a} & -b & -abc & -c \\ 0 & 0 & 0 & 0 & 1 & ac & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & ab & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & \frac{1}{a} & 0 & 2b & ab^2 & 0 \\ c & \frac{1}{a} & 0 & 0 & 0 & ac^2 & 2c \\ 0 & 0 & 0 & \frac{1}{a} & -c & -abc & -b \\ 0 & 0 & 0 & 0 & 0 & ac & 1 \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 1 & ab & 0 \end{pmatrix} \middle| \; a \in \mathbb{C}^*,\, b, c \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 3.2.1 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & \frac{1}{\alpha} & 0 & 0 & 0 & \alpha\beta^2 & 2\beta \\ \gamma & 0 & \frac{1}{\alpha} & 0 & 2\gamma & \alpha\gamma^2 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} & -\beta & -\alpha\beta\gamma & -\gamma \\ 0 & 0 & 0 & 0 & 1 & \alpha\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha\beta & 1 \end{pmatrix}, \text{ or} \right.$$

$$\left. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & 0 & \frac{1}{\alpha} & 0 & 2\beta & \alpha\beta^2 & 0 \\ \gamma & \frac{1}{\alpha} & 0 & 0 & 0 & \alpha\gamma^2 & 2\gamma \\ 0 & 0 & 0 & \frac{1}{\alpha} & -\gamma & -\alpha\beta\gamma & -\beta \\ 0 & 0 & 0 & 0 & 0 & \alpha\gamma & 1 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 1 & \alpha\beta & 0 \end{pmatrix} \middle| \; I\overline{I} = E,\ \alpha \in \mathbb{C}^*,\ \beta, \gamma \in \mathbb{C} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 3.2.1 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, b, c, x, y, z, t)$, $I_2(X) = (\overline{a}, \overline{c}, \overline{b}, \overline{x}, \overline{t}, \overline{z}, \overline{y})$, $I_2(X) = X$. It follows that $X = (a, b, \overline{b}, x, y, z, \overline{y})$, where $b, y \in \mathbb{C}$, $a, x, z \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$e_1' = e_1, \quad e_2' = e_2 + e_3,$$

$$e_3' = ie_2 - ie_3,$$

$$u_1' = u_1, \quad u_2' = \frac{1}{2}u_2 + \frac{1}{2}u_4,$$

$$u_3' = \frac{1}{2}u_3, \quad u_4' = \frac{i}{2}u_2 - \frac{i}{2}u_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $3.2^2.1$.

Proposition 3.2.2. *Any real form of the pair* 3.2.2 *is equivalent to one and only one pair:*

$$3.2^1.2.$$

Proof. 1. The group of automorphisms of the pair 3.2.2 has the form:

$$\mathcal{A} = \left\{ \left. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & \frac{1}{a} & 0 & 0 & 0 & ab^2 & 2b \\ c & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & -adb & -abc & -c \\ 0 & 0 & 0 & 0 & ad & ac & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & ab & 1 \end{pmatrix} \right| a, d \in \mathbb{C}^*,\, b, c \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 3.2.2 has the form:

$$\mathcal{I} = \left\{ I = \left. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & \frac{1}{\alpha} & 0 & 0 & 0 & \alpha\beta^2 & 2\beta \\ \gamma & 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & -\alpha\delta\beta & -\alpha\beta\gamma & -\gamma \\ 0 & 0 & 0 & 0 & \alpha\delta & \alpha\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha\beta & 1 \end{pmatrix} \right| \begin{matrix} I\overline{I} = E,\ \alpha, \delta \in \mathbb{C}^*, \\ \beta, \gamma \in \mathbb{C} \end{matrix} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 3.2.2 is conjugate to one and only one:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 3.2.3. *Any real form of the pair* 3.2.3 *is equivalent to one and only one pair:*

$$3.2^1.3.$$

Proof. 1. The group of automorphisms of the pair 3.2.3 has the form:

$$\mathcal{A} = \left\{ \left. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a^2d & 0 & 0 & 0 & 0 & 0 \\ c & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^3d^2 & -db & -\frac{bc}{2} & -\frac{a^3dc}{2} \\ 0 & 0 & 0 & 0 & ad & \frac{ac}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & b & a^3d \end{pmatrix} \right| a, d \in \mathbb{C}^*,\, b, c \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 3.2.3 has the form:

$$\mathcal{I} = \left\{ I = \left. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha^2\delta & 0 & 0 & 0 & 0 & 0 \\ \gamma & 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha^3\delta^2 & -\delta\beta & -\frac{\beta\gamma}{2} & -\frac{\alpha^3\delta\gamma}{2} \\ 0 & 0 & 0 & 0 & \alpha\delta & \frac{\alpha\gamma}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & \alpha^3\delta \end{pmatrix} \right| \begin{matrix} I\overline{I} = E,\ \alpha, \delta \in \mathbb{C}^*, \\ \beta, \gamma \in \mathbb{C} \end{matrix} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 3.2.3 is conjugate to one and only one:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 3.2.4. *Any real form of the pair* 3.2.4 *is equivalent to one and only one of the following pairs:*

$$3.2^1.4\ (\operatorname{Re}\lambda \geqslant 0,\ \operatorname{Im}\lambda = 0), \quad 3.2^2.2\ (\operatorname{Re}\lambda = 0,\ \operatorname{Im}\lambda \geqslant 0).$$

Proof. Consider the following cases:

1°. $\lambda = 0$.

1. The group of automorphisms of the pair 3.2.4 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & d & 0 & 0 & 0 & 0 & 0 \\ c & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & adf & -abf & -abc & -acd \\ 0 & 0 & 0 & 0 & af & ac & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & ab & ad \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & f & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & adf & -acf & -abc & -abd \\ 0 & 0 & 0 & 0 & 0 & ac & ad \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & af & ab & 0 \end{pmatrix} \middle| \; a, d, f \in \mathbb{C}^*,\ b, c \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 3.2.4 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & \delta & 0 & 0 & 0 & 0 & 0 \\ \gamma & 0 & \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha\delta\varepsilon & -\alpha\beta\varepsilon & -\alpha\beta\gamma & -\alpha\gamma\delta \\ 0 & 0 & 0 & 0 & \alpha\varepsilon & \alpha\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha\beta & \alpha\delta \end{pmatrix}, \text{ or} \right.$$
$$\left. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & 0 & \varepsilon & 0 & 0 & 0 & 0 \\ \gamma & \delta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha\delta\varepsilon & -\alpha\gamma\varepsilon & -\alpha\beta\gamma & -\alpha\beta\delta \\ 0 & 0 & 0 & 0 & 0 & \alpha\gamma & \alpha\delta \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & \alpha\varepsilon & \alpha\beta & 0 \end{pmatrix} \middle| \; \begin{matrix} I\overline{I} = E,\ \alpha, \delta, \varepsilon \in \mathbb{C}^*, \\ \beta, \gamma \in \mathbb{C} \end{matrix} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 3.2.4 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, b, c, x, y, z, t)$, $I_2(X) = (\overline{a}, \overline{c}, \overline{b}, \overline{x}, \overline{t}, \overline{z}, \overline{y})$, $I_2(X) = X$. It follows that $X = (a, b, \overline{b}, x, y, z, \overline{y})$, where $b, y \in \mathbb{C}$, $a, x, z \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$e_1' = e_1, \quad e_2' = e_2 + e_3,$$

$$e_3' = ie_2 - ie_3,$$

$$u_1' = u_1, \quad u_2' = \frac{1}{2}u_2 + \frac{1}{2}u_4,$$
$$u_3' = \frac{1}{2}u_3, \quad u_4' = \frac{i}{2}u_2 - \frac{i}{2}u_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $3.2^2.2$ ($\lambda = 0$).

2°. $\lambda = 1$.

1. The group of automorphisms of the pair 3.2.4 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 \\ c & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & abd & -df & -\frac{cf}{2} & -\frac{abc}{2} \\ 0 & 0 & 0 & 0 & ad & \frac{ac}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & f & ab \end{pmatrix} \middle| \; a, b, d \in \mathbb{C}^*, \, c, f \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 3.2.4 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ \gamma & 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha\beta\delta & -\delta\varepsilon & -\frac{\varepsilon\gamma}{2} & -\frac{\alpha\beta\gamma}{2} \\ 0 & 0 & 0 & 0 & \alpha\delta & \frac{\alpha\gamma}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon & \alpha\beta \end{pmatrix} \middle| \; \begin{array}{c} I\overline{I} = E, \; \alpha, \beta, \delta \in \mathbb{C}^*, \\ \gamma, \varepsilon \in \mathbb{C} \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 3.2.4 is conjugate to one and only one:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3°. $\operatorname{Re}\lambda > 0,\ \operatorname{Im}\lambda = 0,\ \lambda \neq 1$.

1. The group of automorphisms of the pair 3.2.4 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & f & 0 & 0 & 0 & 0 & 0 \\ c & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & adf & -\frac{adb}{1-\lambda} & -\frac{abc}{1-\lambda^2} & -\frac{acf}{1+\lambda} \\ 0 & 0 & 0 & 0 & ad & \frac{ac}{1+\lambda} & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{ab}{1-\lambda} & af \end{pmatrix} \middle| \ a, d, f \in \mathbb{C}^*,\ b, c \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 3.2.4 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & \varepsilon & 0 & 0 & 0 & 0 & 0 \\ \gamma & 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha\delta\varepsilon & -\frac{\alpha\delta\beta}{1-\lambda} & -\frac{\alpha\beta\gamma}{1-\lambda^2} & -\frac{\alpha\gamma\varepsilon}{1+\lambda} \\ 0 & 0 & 0 & 0 & \alpha\delta & \frac{\alpha\gamma}{1+\lambda} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\alpha\beta}{1-\lambda} & \alpha\varepsilon \end{pmatrix} \middle| \ \begin{matrix} I\overline{I} = E,\ \alpha, \delta, \varepsilon \in \mathbb{C}^*, \\ \beta, \gamma \in \mathbb{C} \end{matrix} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 3.2.4 is conjugate to one and only one:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4°. $\operatorname{Re}\lambda = 0,\ \operatorname{Im} > 0$.

1. The group of automorphisms of the pair 3.2.4 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & f & 0 & 0 & 0 & 0 & 0 \\ c & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & adf & -\frac{adb}{1-\lambda} & -\frac{abc}{1-\lambda^2} & -\frac{acf}{1+\lambda} \\ 0 & 0 & 0 & 0 & ad & \frac{ac}{1+\lambda} & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{ab}{1-\lambda} & af \end{pmatrix} \middle| \ a, d, f \in \mathbb{C}^*,\ b, c \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 3.2.4 has the form:

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & 0 & \delta & 0 & 0 & 0 & 0 \\ \gamma & \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha\delta\varepsilon & -\frac{\alpha\delta\gamma}{1+\lambda} & -\frac{\alpha\beta\gamma}{1-\lambda^2} & -\frac{\alpha\beta\varepsilon}{1-\lambda} \\ 0 & 0 & 0 & 0 & 0 & \frac{\alpha\gamma}{1+\lambda} & \alpha\varepsilon \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & \alpha\delta & \frac{\alpha\beta}{1-\lambda} & 0 \end{pmatrix} \;\middle|\; \begin{array}{c} I\overline{I} = E,\ \alpha, \delta, \varepsilon \in \mathbb{C}^*, \\ \beta, \gamma \in \mathbb{C} \end{array} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 3.2.4 is conjugate to one and only one:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_1.

Let $X = (a, b, c, x, y, z, t)$, $I_1(X) = (\overline{a}, \overline{c}, \overline{b}, \overline{x}, \overline{t}, \overline{z}, \overline{y})$, $I_1(X) = X$. It follows that $X = (a, b, \overline{b}, x, y, z, \overline{y})$, where $b, y \in \mathbb{C}$, $a, x, z \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$e_1' = e_1, \quad e_2' = e_2 + e_3,$$
$$e_3' = ie_2 - ie_3,$$
$$u_1' = u_1, \qquad u_2' = \frac{1}{2}u_2 + \frac{1}{2}u_4,$$
$$u_3' = \frac{1}{2}u_3, \quad u_4' = \frac{i}{2}u_2 - \frac{i}{2}u_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_1 has the form $3.2^2.2$ $(\lambda > 0)$.

For other values of λ there exist no anti–involutions.

Theorem 3.3. *Any real form of the linear Lie algebra* 3.3 *is conjugate to one and only one of the following linear Lie algebras:*

$$3.3^1 \quad \begin{pmatrix} 0 & y & 0 & z \\ 0 & x & -z & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -y & -x \end{pmatrix} \qquad 3.3^2 \quad \begin{pmatrix} 0 & y & 0 & -z \\ 0 & 0 & -y & -x \\ 0 & 0 & 0 & 0 \\ 0 & x & z & 0 \end{pmatrix}$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 3.3^1 *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	u_1	0
u_2	$-u_2$	$-u_1$	0	0	0	pe_3+u_2	0
u_3	0	u_4	u_2	$-u_1$	$-pe_3-u_2$	0	$-pe_2-u_4$
u_4	u_4	0	$-u_1$	0	0	pe_2+u_4	0

2.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	e_3	0
u_3	0	u_4	u_2	0	$-e_3$	0	$-e_2$
u_4	u_4	0	$-u_1$	0	0	e_2	0

3.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	$-e_3$	0
u_3	0	u_4	u_2	0	e_3	0	e_2
u_4	u_4	0	$-u_1$	0	0	$-e_2$	0

4.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	0	0
u_3	0	u_4	u_2	0	0	0	0
u_4	u_4	0	$-u_1$	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 3.3^2 *is equivalent to one and only one of the following pairs:*

1.

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	0	u_4	0	$-u_2$
e_2	e_3	0	0	0	u_1	$-u_2$	0
e_3	$-e_2$	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	u_1	0
u_2	$-u_4$	$-u_1$	0	0	0	pe_2+u_2	0
u_3	0	u_2	$-u_4$	$-u_1$	$-pe_2-u_2$	0	pe_3-u_4
u_4	u_2	0	u_1	0	0	$-pe_3+u_4$	0

2.

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	0	u_4	0	$-u_2$
e_2	e_3	0	0	0	u_1	$-u_2$	0
e_3	$-e_2$	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0	0
u_2	$-u_4$	$-u_1$	0	0	0	e_2	0
u_3	0	u_2	$-u_4$	0	$-e_2$	0	e_3
u_4	u_2	0	u_1	0	0	$-e_3$	0

3.

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	0	u_4	0	$-u_2$
e_2	e_3	0	0	0	u_1	$-u_2$	0
e_3	$-e_2$	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0	0
u_2	$-u_4$	$-u_1$	0	0	0	$-e_2$	0
u_3	0	u_2	$-u_4$	0	e_2	0	$-e_3$
u_4	u_2	0	u_1	0	0	e_3	0

4.

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	0	u_4	0	$-u_2$
e_2	e_3	0	0	0	u_1	$-u_2$	0
e_3	$-e_2$	0	0	0	0	u_4	$-u_1$
u_1	0	0	0	0	0	0	0
u_2	$-u_4$	$-u_1$	0	0	0	0	0
u_3	0	u_2	$-u_4$	0	0	0	0
u_4	u_2	0	u_1	0	0	0	0

The proof of the Theorem follows from Propositions 3.3.1–3.3.3.

Proposition 3.3.1. *Any real form of the pair* 3.3.1 *is equivalent to one and only one of the following pairs:*

$$3.3^1.1, \quad 3.3^2.1.$$

Proof. 1. The group of automorphisms of the pair 3.3.1 has the form:

$$\mathcal{A}=\left\{\begin{pmatrix}1&0&0&0&0&0&0\\ b&d&0&0&0&0&0\\ c&0&f&0&0&0&0\\ 0&0&0&df&bf&a&-cd\\ 0&0&0&0&f&c&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&-b&d\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}-1&0&0&0&0&0&0\\ b&0&f&0&0&0&0\\ c&d&0&0&0&0&0\\ 0&0&0&df&cf&a&-bd\\ 0&0&0&0&0&-c&d\\ 0&0&0&0&0&1&0\\ 0&0&0&0&f&b&0\end{pmatrix}\;\middle|\; d,f\in\mathbb{C}^*,\ a,b,c\in\mathbb{C}\right\}.$$

2. The set of anti–involutions of the pair 3.3.1 has the form:
if $p\in\mathbb{R}$

$$\mathcal{I}=\left\{I=\begin{pmatrix}1&0&0&0&0&0&0\\ \beta&\delta&0&0&0&0&0\\ \gamma&0&\varepsilon&0&0&0&0\\ 0&0&0&\delta\varepsilon&\beta\varepsilon&\alpha&-\gamma\delta\\ 0&0&0&0&\varepsilon&\gamma&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&-\beta&\delta\end{pmatrix},\text{ or}\right.$$
$$\left.\begin{pmatrix}-1&0&0&0&0&0&0\\ \beta&0&\varepsilon&0&0&0&0\\ \gamma&\delta&0&0&0&0&0\\ 0&0&0&\delta\varepsilon&\gamma\varepsilon&\alpha&-\beta\delta\\ 0&0&0&0&0&-\gamma&\delta\\ 0&0&0&0&0&1&0\\ 0&0&0&0&\varepsilon&\beta&0\end{pmatrix}\;\middle|\; I\bar{I}=E,\ \delta,\varepsilon\in\mathbb{C}^*,\ \alpha,\beta,\gamma\in\mathbb{C}\right\}.$$

For other values of p there exist no anti–involutions.

3. Let $A\in\mathcal{A}$, $I\in\mathcal{I}$. Using the operation $I\mapsto AI\bar{A}^{-1}$ we obtain that any anti–involution of the pair 3.3.1 is conjugate to one and only one of the following:

$$I_1=\begin{pmatrix}1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\end{pmatrix},\qquad I_2=\begin{pmatrix}-1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a, b, c, x, y, z, t)$, $I_2(X) = (-\overline{a}, \overline{c}, \overline{b}, \overline{x}, \overline{t}, \overline{z}, \overline{y})$, $I_2(X) = X$. It follows that $X = (a, b, \overline{b}, x, y, z, \overline{y})$, where $b, y \in \mathbb{C}$, $a \in \mathbb{R}i$, $x, z \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$e_1' = ie_1, \quad e_2' = e_2 + e_3,$$

$$e_3' = ie_2 - ie_3,$$

$$u_1' = 2u_1, \quad u_2' = u_2 + u_4,$$
$$u_3' = u_3, \quad u_4' = iu_2 - iu_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form 3.3^2.1.

Proposition 3.3.2. *Any real form of the pair* 3.3.2 *is equivalent to one and only one of the following pairs:*

$$3.3^1.2, \quad 3.3^1.3, \quad 3.3^2.2, \quad 3.3^2.3.$$

Proof. 1. The group of automorphisms of the pair 3.3.2 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & d & 0 & 0 & 0 & 0 & 0 \\ c & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & df & bf & a & -cd \\ 0 & 0 & 0 & 0 & f & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b & d \end{pmatrix}, \right.$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & d & 0 & 0 & 0 & 0 & 0 \\ c & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -df & -bf & a & cd \\ 0 & 0 & 0 & 0 & -f & -c & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & -d \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & f & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & df & cf & a & -bd \\ 0 & 0 & 0 & 0 & 0 & -c & d \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & f & b & 0 \end{pmatrix},$$
$$\left. \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & f & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -df & -cf & a & bd \\ 0 & 0 & 0 & 0 & 0 & c & -d \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -f & -b & 0 \end{pmatrix} \middle| \; d, f \in \mathbb{C}^*, \; a, b, c \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 3.3.2 has the form:

$$\mathcal{I}=\left\{ I \in \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & \delta & 0 & 0 & 0 & 0 & 0 \\ \gamma & 0 & \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta\varepsilon & \beta\varepsilon & \alpha & -\gamma\delta \\ 0 & 0 & 0 & 0 & \varepsilon & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta & \delta \end{pmatrix}, \right.\right.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & \delta & 0 & 0 & 0 & 0 & 0 \\ \gamma & 0 & \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta\varepsilon & -\beta\varepsilon & \alpha & \gamma\delta \\ 0 & 0 & 0 & 0 & -\varepsilon & -\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & -\delta \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & 0 & \varepsilon & 0 & 0 & 0 & 0 \\ \gamma & \delta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta\varepsilon & \gamma\varepsilon & \alpha & -\beta\delta \\ 0 & 0 & 0 & 0 & 0 & -\gamma & \delta \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon & \beta & 0 \end{pmatrix},$$

$$\left.\left. \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & 0 & \varepsilon & 0 & 0 & 0 & 0 \\ \gamma & \delta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta\varepsilon & -\gamma\varepsilon & \alpha & \beta\delta \\ 0 & 0 & 0 & 0 & 0 & \gamma & -\delta \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -\varepsilon & -\beta & 0 \end{pmatrix} \right\} \;\middle|\; I\overline{I}=E,\ \delta,\varepsilon\in\mathbb{C}^*,\ \alpha,\beta,\gamma\in\mathbb{C} \right\}.$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 3.3.2 is conjugate to one and only one of the following:

$$I_1=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_2=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$I_3=\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad I_4=\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a,b,c,x,y,z,t)$, $I_2(X) = (\overline{a},\overline{b},\overline{c},-\overline{x},-\overline{y},-\overline{z},-\overline{t})$, $I_2(X) = X$. It follows that $X = (a,b,c,x,y,z,t)$, where $x,y,z,t \in \mathbb{R}i$, $a,b,c \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$e_1' = e_1, \quad e_2' = e_2, \quad e_3' = e_3,$$

$$u'_1 = iu_1, \quad u'_2 = iu_2,$$
$$u'_3 = iu_3, \quad u'_4 = iu_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $3.3^1.3$.

5. Find the set of fixed points of the mapping I_3.

Let $X = (a, b, c, x, y, z, t)$, $I_3(X) = (-\overline{a}, \overline{c}, \overline{b}, \overline{x}, \overline{t}, \overline{z}, \overline{y})$, $I_3(X) = X$. It follows that $X = (a, b, \overline{b}, x, y, z, \overline{y})$, where $b, y \in \mathbb{C}$, $a \in \mathbb{R}i$, $x, z \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$e'_1 = ie_1, \quad e'_2 = e_2 + e_3,$$
$$e'_3 = ie_2 - ie_3,$$
$$u'_1 = 2u_1, \quad u'_2 = u_2 + u_4,$$
$$u'_3 = u_3, \quad u'_4 = iu_2 - iu_4.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_3 has the form $3.3^2.2$.

6. Find the set of fixed points of the mapping I_4.

Let $X = (a, b, c, x, y, z, t)$, $I_4(X) = (-\overline{a}, \overline{c}, \overline{b}, -\overline{x}, -\overline{t}, -\overline{z}, -\overline{y})$, $I_4(X) = X$. It follows that $X = (a, b, \overline{b}, x, y, z, -\overline{y})$, where $b, y \in \mathbb{C}$, $a, x, z \in \mathbb{R}i$.

Consider the following basis of the set of fixed points:

$$e'_1 = ie_1, \quad e'_2 = e_2 + e_3,$$
$$e'_3 = ie_2 - ie_3,$$
$$u'_1 = 2iu_1, \quad u'_2 = iu_2 + iu_4,$$
$$u'_3 = iu_3, \quad u'_4 = u_4 - u_2.$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_4 has the form $3.3^2.3$.

Proposition 3.3.3. *Any real form of the pair* 3.3.3 *is equivalent to one and only one of the following pairs:*

$$3.3^1.4, \quad 3.3^2.4.$$

Proof. 1. The group of automorphisms of the pair 3.3.3 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & d & 0 & 0 & 0 & 0 & 0 \\ c & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & adf & abf & l & -acd \\ 0 & 0 & 0 & 0 & af & ac & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & -ab & ad \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & f & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & adf & acf & l & -abd \\ 0 & 0 & 0 & 0 & 0 & -ac & ad \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & af & ab & 0 \end{pmatrix} \middle| \; a, d, f \in \mathbb{C}^*, \, b, c, l \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 3.3.3 has the form:

$$
\mathcal{I}=\left\{ I=\begin{pmatrix}1&0&0&0&0&0&0\\ \beta&\delta&0&0&0&0&0\\ \gamma&0&\varepsilon&0&0&0&0\\ 0&0&0&\alpha\delta\varepsilon&\alpha\beta\varepsilon&\eta&-\alpha\gamma\delta\\ 0&0&0&0&\alpha\varepsilon&\alpha\gamma&0\\ 0&0&0&0&0&\alpha&0\\ 0&0&0&0&0&-\alpha\beta&\alpha\delta\end{pmatrix}, \text{ or } \right.
$$

$$
\left. \begin{pmatrix}-1&0&0&0&0&0&0\\ \beta&0&\varepsilon&0&0&0&0\\ \gamma&\delta&0&0&0&0&0\\ 0&0&0&\alpha\delta\varepsilon&\alpha\gamma\varepsilon&\eta&-\alpha\beta\delta\\ 0&0&0&0&0&-\alpha\gamma&\alpha\delta\\ 0&0&0&0&0&\alpha&0\\ 0&0&0&0&\alpha\varepsilon&\alpha\beta&0\end{pmatrix} \;\middle|\; \begin{array}{c} I\overline{I}=E,\ \alpha,\delta,\varepsilon\in\mathbb{C}^*,\\ \beta,\gamma,\eta\in\mathbb{C}\end{array} \right\}.
$$

3. Let $A \in \mathcal{A}$, $I \in \mathcal{I}$. Using the operation $I \mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 3.3.3 is conjugate to one and only one of the following:

$$
I_1=\begin{pmatrix}1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\end{pmatrix},\qquad I_2=\begin{pmatrix}-1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\end{pmatrix}.
$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a,b,c,x,y,z,t)$, $I_2(X) = (-\overline{a},\overline{c},\overline{b},\overline{x},\overline{t},\overline{z},\overline{y})$, $I_2(X) = X$. It follows that $X = (a,b,\overline{b},x,y,z,\overline{y})$, where $b,y \in \mathbb{C}$, $a \in \mathbb{R}i$, $x,z \in \mathbb{R}$.

Consider the following basis of the set of fixed points:

$$
\begin{gathered}
e_1' = ie_1, \quad e_2' = e_2 + e_3,\\
e_3' = ie_2 - ie_3,\\
u_1' = 2u_1, \quad u_2' = u_2 + u_4,\\
u_3' = u_3, \quad u_4' = iu_2 - iu_4.
\end{gathered}
$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $3.3^2.4$.

Theorem 4.1. *Any real form of the linear Lie algebra 4.1 is conjugate to one and only one of the following linear Lie algebras:*

$$
4.1^1\quad \begin{pmatrix}x&z&0&t\\ 0&y&-t&0\\ 0&0&-x&0\\ 0&0&-z&-y\end{pmatrix}\qquad 4.1^2\quad \begin{pmatrix}x&z&0&-t\\ 0&0&-z&-y\\ 0&0&-x&0\\ 0&y&t&0\end{pmatrix}
$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ of type 4.1^1 *is equivalent to one and only one pair:*

1.

[,]	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	e_3	e_4	u_1	0	$-u_3$	0
e_2	0	0	$-e_3$	e_4	0	u_2	0	$-u_4$
e_3	$-e_3$	e_3	0	0	0	u_1	$-u_4$	0
e_4	$-e_4$	$-e_4$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	0	0	0
u_2	0	$-u_2$	$-u_1$	0	0	0	0	0
u_3	u_3	0	u_4	u_2	0	0	0	0
u_4	0	u_4	0	$-u_1$	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ of type 4.1^2 *is equivalent to one and only one pair:*

1.

[,]	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	e_3	e_4	u_1	0	$-u_3$	0
e_2	0	0	$-e_4$	e_3	0	u_4	0	$-u_2$
e_3	$-e_3$	e_4	0	0	0	u_1	$-u_2$	0
e_4	$-e_4$	$-e_3$	0	0	0	0	u_4	$-u_1$
u_1	$-u_1$	0	0	0	0	0	0	0
u_2	0	$-u_4$	$-u_1$	0	0	0	0	0
u_3	u_3	0	u_2	$-u_4$	0	0	0	0
u_4	0	u_2	0	u_1	0	0	0	0

Proof. 1. The group of automorphisms of the pair 4.1.1 has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & -b & d & 0 & 0 & 0 & 0 & 0 \\ c & c & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & adf & -abf & -abc & -acd \\ 0 & 0 & 0 & 0 & 0 & af & ac & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ab & ad \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & b & 0 & f & 0 & 0 & 0 & 0 \\ c & -c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & adf & -acf & -abc & -abd \\ 0 & 0 & 0 & 0 & 0 & 0 & ac & ad \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & af & ab & 0 \end{pmatrix} \;\middle|\; a, d, f \in \mathbb{C}^*,\ b, c \in \mathbb{C} \right\}.$$

2. The set of anti–involutions of the pair 4.1.1 has the form:

$$\mathcal{I}=\left\{I=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\\beta&-\beta&\delta&0&0&0&0&0\\\gamma&\gamma&0&\varepsilon&0&0&0&0\\0&0&0&0&\alpha\delta\varepsilon&-\alpha\beta\varepsilon&-\alpha\beta\gamma&-\alpha\gamma\delta\\0&0&0&0&0&\alpha\varepsilon&\alpha\gamma&0\\0&0&0&0&0&0&\alpha&0\\0&0&0&0&0&0&\alpha\beta&\alpha\delta\end{pmatrix},\text{ or}\right.$$

$$\left.\begin{pmatrix}1&0&0&0&0&0&0&0\\0&-1&0&0&0&0&0&0\\\beta&\beta&0&\varepsilon&0&0&0&0\\\gamma&-\gamma&\delta&0&0&0&0&0\\0&0&0&0&\alpha\delta\varepsilon&-\alpha\gamma\varepsilon&-\alpha\beta\gamma&-\alpha\beta\delta\\0&0&0&0&0&0&\alpha\gamma&\alpha\delta\\0&0&0&0&0&0&\alpha&0\\0&0&0&0&0&\alpha\varepsilon&\alpha\beta&0\end{pmatrix}\;\middle|\;\begin{array}{c}I\overline{I}=E,\ \alpha,\delta,\varepsilon\in\mathbb{C}^*,\\ \beta,\gamma\in\mathbb{C}\end{array}\right\}.$$

3. Let $A\in\mathcal{A}$, $I\in\mathcal{I}$. Using the operation $I\mapsto AI\overline{A}^{-1}$ we obtain that any anti–involution of the pair 4.1.1 is conjugate to one and only one of the following:

$$I_1=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix},\qquad I_2=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&-1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0\end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X=(a,b,c,d,x,y,z,t)$, $I_2(X)=(\overline{a},-\overline{b},\overline{d},\overline{c},\overline{x},\overline{t},\overline{z},\overline{y})$, $I_2(X)=X$. It follows that $X=(a,b.c,\overline{c},x,y,z,\overline{y})$, where $c,y\in\mathbb{C}$, $b\in\mathbb{R}i$, $a,x,z\in\mathbb{R}$.

Consider the following basis of the set of fixed points:

$$\begin{aligned}e_1'&=e_1, & e_2'&=ie_2,\\ e_3'&=e_3+e_4, & e_4'&=ie_3-ie_4,\\ u_1'&=2u_1, & u_2'&=u_2+u_4,\\ u_3'&=u_3, & u_4'&=iu_2-iu_4.\end{aligned}$$

In this basis the multiplication table of the real form corresponding to the anti–involution I_2 has the form $4.1^2.1$.

4. Real forms of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ with non-solvable subalgebra $\mathfrak{g}$

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a complex isotropically faithful pair, where $\mathfrak{g}$ a non-solvable subalgebra of $\bar{\mathfrak{g}}$.

In this case the computation of the automorphism group and of the set of anti-involution for the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ becomes increasingly complicated, especially by in the non-trivial case.

We shall act in the following way:

1) We find all real forms $\mathfrak{g}^\sigma$ of the linear Lie algebra $\mathfrak{g}$.

2) For each real linear Lie algebra $\mathfrak{g}^\sigma$ we classify the isotropically faithful pairs $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$.

1. Finding of real forms of the linear Lie algebra $\mathfrak{g}$.

Consider the following set:

$$\mathrm{Conj}(\mathfrak{g}) = \{A \in \mathrm{GL}(4, \mathbb{C}) | A\mathfrak{g}A^{-1} = \mathfrak{g}\}.$$

Let $\mathcal{E} = \{e_1, ..., e_n\}$ be a basis of $\mathfrak{g}$, $A \in \mathrm{Conj}(\mathfrak{g})$.

Then the mapping $\pi : \mathfrak{g} \to \mathfrak{g}$ defined by $\pi(X) = AXA^{-1}$ is an isomorphism of $\mathfrak{g}$.

Consider the following set:

$$\mathcal{P} = \{P \in \mathrm{GL}(4, \mathbb{C}) | P\overline{XP} \in \mathfrak{g}, \text{ for all } X \in \mathfrak{g},\ P\overline{P} = E\}.$$

Then the mapping $\sigma : \mathfrak{g} \to \mathfrak{g}$ defined by $\sigma(X) = PX\overline{P}$ is an anti-involution of $\mathfrak{g}$.

Note that the set of anti-involutions of $\mathfrak{g}$ is completely determined by $\mathcal{P}$, and the group of automorphisms of $\mathfrak{g}$ is completely determined by $\mathrm{Conj}(\mathfrak{g})$.

It follows that in order to obtain all anti-involutions up to the operation $\pi\sigma\pi^{-1}$, it suffices to describe all $P \in \mathcal{P}$ up to the operation $AP\overline{A}^{-1}$, where $A \in \mathrm{Conj}(\mathfrak{g})$.

We use the following algorithm of finding all real forms of the linear Lie algebra $\mathfrak{g}$:

1) We find $\mathrm{Conj}(\mathfrak{g})$.

2) We find $\mathcal{P}$.

3) Let $A \in \mathrm{Conj}(\mathfrak{g})$, $P \in \mathcal{P}$. Using the operation $P \mapsto AP\overline{A}^{-1}$ we find $P_1, P_2, ..., P_k$ such that $P_i \neq AP_j\overline{A}^{-1}$ for all $A \in \mathrm{Conj}(\mathfrak{g})$, $i, j = \overline{1, k}$, $i \neq j$.

4) For each matrix P_j, $j = \overline{1, k}$ we find the matrix I_j corresponding to the anti-involution σ_j.

5)For each anti-involution I_j, $j = \overline{1, k}$ we find the correspondihg real form $\mathfrak{g}^\sigma$ of $\mathfrak{g}$.

2. Classification of isotropically faithful pairs $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ corresponding to the linear Lie algebra $\mathfrak{g}^\sigma$.

The method of finding real isotropically faithful pairs is similar to that used for finding complex pairs (for a detailed description see [KT] or [K1]).

For example, we prove the following Theorem:

Theorem 4.2. *Any real form of the linear Lie algebra 4.2 is conjugate to one and only one of the following linear Lie algebras:*

$$4.2^1 \quad \begin{pmatrix} x & y & 0 & 0 \\ z & t & 0 & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & -t \end{pmatrix} \qquad 4.2^2 \quad \begin{pmatrix} 0 & y & -x & -z \\ -y & 0 & -z & -t \\ x & z & 0 & y \\ z & t & -y & 0 \end{pmatrix}$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 4.2^1 *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	0	e_1+3e_2	$2e_3$
u_2	u_2	$-u_2$	$-u_1$	0	0	0	$2e_4$	$-e_1+3e_2$
u_3	u_3	u_3	u_4	0	$-e_1-3e_2$	$-2e_4$	0	0
u_4	$-u_4$	u_4	0	u_3	$-2e_3$	e_1-3e_2	0	0

2.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	0	$-e_1-3e_2$	$-2e_3$
u_2	u_2	$-u_2$	$-u_1$	0	0	0	$-2e_4$	e_1-3e_2
u_3	u_3	u_3	u_4	0	e_1+3e_2	$2e_4$	0	0
u_4	$-u_4$	u_4	0	u_3	$2e_3$	$3e_2-e_1$	0	0

3.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	0	0	0
u_2	u_2	$-u_2$	$-u_1$	0	0	0	0	0
u_3	u_3	u_3	u_4	0	0	0	0	0
u_4	$-u_4$	u_4	0	u_3	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 4.2^2 *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_4$	$-2e_3$	u_3	$-u_4$	$-u_1$	u_2
e_2	0	0	0	0	u_3	u_4	$-u_1$	$-u_2$
e_3	$-2e_4$	0	0	$2e_1$	$-u_2$	u_1	$-u_4$	u_3
e_4	$2e_3$	0	$-2e_1$	0	u_4	u_3	$-u_2$	$-u_1$
u_1	$-u_3$	$-u_3$	u_2	$-u_4$	0	$-e_3$	e_1+3e_2	e_4
u_2	u_4	$-u_4$	$-u_1$	$-u_3$	e_3	0	e_4	$-e_1+3e_2$
u_3	u_1	u_1	u_4	u_2	$-e_1-3e_2$	$-e_4$	0	$-e_3$
u_4	$-u_2$	u_2	$-u_3$	u_1	$-e_4$	e_1-3e_2	e_3	0

2.

$[,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_4$	$-2e_3$	u_3	$-u_4$	$-u_1$	u_2
e_2	0	0	0	0	u_3	u_4	$-u_1$	$-u_2$
e_3	$-2e_4$	0	0	$2e_1$	$-u_2$	u_1	$-u_4$	u_3
e_4	$2e_3$	0	$-2e_1$	0	u_4	u_3	$-u_2$	$-u_1$
u_1	$-u_3$	$-u_3$	u_2	$-u_4$	0	e_3	$-e_1-3e_2$	$-e_4$
u_2	u_4	$-u_4$	$-u_1$	$-u_3$	$-e_3$	0	$-e_4$	e_1-3e_2
u_3	u_1	u_1	u_4	u_2	e_1+3e_2	e_4	0	e_3
u_4	$-u_2$	u_2	$-u_3$	u_1	e_4	$3e_2-e_1$	$-e_3$	0

3.

$[,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_4$	$-2e_3$	u_3	$-u_4$	$-u_1$	u_2
e_2	0	0	0	0	u_3	u_4	$-u_1$	$-u_2$
e_3	$-2e_4$	0	0	$2e_1$	$-u_2$	u_1	$-u_4$	u_3
e_4	$2e_3$	0	$-2e_1$	0	u_4	u_3	$-u_2$	$-u_1$
u_1	$-u_3$	$-u_3$	u_2	$-u_4$	0	0	0	0
u_2	u_4	$-u_4$	$-u_1$	$-u_3$	0	0	0	0
u_3	u_1	u_1	u_4	u_2	0	0	0	0
u_4	$-u_2$	u_2	$-u_3$	u_1	0	0	0	0

The proof of the Theorem follows from Propositions 1,2.

Proposition 1. *Any real form of the linear Lie algebra* 4.2 *is conjugate to one and only one of the following linear Lie algebras:* 4.2^1, 4.2^2.

Proof. 1. The group $\mathrm{Conj}(\mathfrak{g})$ has the form:

$$\mathrm{Conj}(\mathfrak{g}) = \left\{ \begin{pmatrix} C & 0 \\ 0 & \alpha^t C^{-1} \end{pmatrix}, \begin{pmatrix} 0 & C \\ \alpha^t C^{-1} & 0 \end{pmatrix} \middle| \; C \in \mathrm{GL}(2,\mathbb{C}),\ \alpha \in \mathbb{C} \right\}.$$

2. The set $\mathcal{P}$ has the form:

$$\mathcal{P} = \left\{ \begin{pmatrix} P_1 & 0 \\ 0 & \beta^t P_1{}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & P_2 \\ \overline{P_2}^{-1} & 0 \end{pmatrix} \middle| \; P_1, P_2 \in \mathrm{GL}(2,\mathbb{C}),\ P_1\overline{P_1} = E,\ {}^tP_2 = \gamma \overline{P_2} \right\}.$$

Here $\beta, \gamma \in \mathbb{C}$, $|\beta| = 1$.

3. Let $A \in \mathrm{Conj}(\mathfrak{g})$, $P \in \mathcal{P}$. Using the operation $P \mapsto AP\overline{A}^{-1}$ we obtain that any matrix $P \in \mathcal{P}$ is conjugate to one and only one of the following:

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

4. For P_j, $j = 1, 2$ we find the matrix I_j. We obtain that any anti-involution of the linear Lie algebra 4.2 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

4. Find the set of fixed points of the mapping I_2.

Let $X = (a,b,c,d)$, $I_2(X) = (-\overline{a}, -\overline{b}, -\overline{d}, -\overline{c})$, $I_2(X) = X$. It follows that $X = (a,b,c,-\overline{c})$, where $a,b \in \mathbb{R}i$, $c \in \mathbb{C}$.

Consider the following basis of the set of fixed points:

$$e_1' = ie_1, \quad e_2' = ie_2, \quad e_3' = e_3 - e_4, \quad e_4' = ie_3 + ie_4.$$

In this basis the real form of $\mathfrak{g}$ corresponding to the anti–involution I_2 has the form 4.2^2.

Proposition 2. *Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ of type 4.2^1 is equivalent to one and only one of the following pairs: $4.2^1.1$, $4.2^1.2$, $4.2^1.3$.*

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ of type 4.2^2 is equivalent to one and only one of the following pairs: $4.2^2.1$, $4.2^2.2$, $4.2^2.3$.

Proof. The case 4.2^1 is over $\mathbb{C}$.

We prove the Proposition by using a similar method of classification over $\mathbb{R}$.

Consider the pairs $4.2^1.1$ and $4.2^1.2$.

Since one of them is isomorphic to $\mathfrak{sl}(3,\mathbb{R})$ and the other to $\mathfrak{su}(2,1)$, we see that the pairs $4.2^1.1$ and $4.2^1.2$ are not equivalent.

Consider the case 4.2^2.

Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 4.2^2 is trivial.*

Proof. Let q be a virtual structure on generalized module 4.2^2. Note that $\mathfrak{a} = \mathbb{R}e_1 \oplus \mathbb{R}e_3 \oplus \mathbb{R}e_4$ is a semisimple subalgebra of the Lie algebra $\mathfrak{g}^\sigma$. Without loss of generality it can be assumed that $q(\mathfrak{a}) = \{0\}$. Therefore

$$C(e_1) = C(e_3) = C(e_4) = 0, \; C(e_2) = (c^1_{ij})_{\substack{1\leqslant i\leqslant 4 \\ 1\leqslant j\leqslant 4}}.$$

Checking condition (3), Chapter I, [K1], we obtain:

$$C(e_1) = C(e_2) = C(e_3) = C(e_4) = 0.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ be a pair of type 4.2^2. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ is defined by the trivial virtual structure.

Then

$$\begin{array}{llll}
[e_1, e_2] = 0, & & & \\
[e_1, e_3] = 2e_4, & [e_2, e_3] = 0, & & \\
[e_1, e_4] = -2e_3, & [e_2, e_4] = 0, & [e_3, e_4] = 2e_1, & \\
[e_1, u_1] = u_3, & [e_2, u_1] = u_3, & [e_3, u_1] = -u_2, & [e_4, u_1] = u_4, \\
[e_1, u_2] = -u_4, & [e_2, u_2] = u_4, & [e_3, u_2] = u_1, & [e_4, u_2] = u_3, \\
[e_1, u_3] = -u_1, & [e_2, u_3] = -u_1, & [e_3, u_3] = -u_4, & [e_4, u_3] = -u_2, \\
[e_1, u_4] = u_2, & [e_2, u_4] = -u_2, & [e_3, u_4] = u_3, & [e_4, u_4] = -u_1.
\end{array}$$

Put

$$\begin{aligned}
[u_1, u_2] &= a_1e_1 + a_2e_2 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3 + \alpha_4u_4, \\
[u_1, u_3] &= b_1e_1 + b_2e_2 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3 + \beta_4u_4, \\
[u_1, u_4] &= c_1e_1 + c_2e_2 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3 + \gamma_4u_4, \\
[u_2, u_3] &= d_1e_1 + d_2e_2 + \delta_1u_1 + \delta_2u_2 + \delta_3u_3 + \delta_4u_4, \\
[u_2, u_4] &= f_1e_1 + f_2e_2 + \eta_1u_1 + \eta_2u_2 + \eta_3u_3 + \eta_4u_4, \\
[u_3, u_4] &= k_1e_1 + k_2e_2 + \varepsilon_1u_1 + \varepsilon_2u_2 + \varepsilon_3u_3 + \varepsilon_4u_4.
\end{aligned}$$

Using the Jacobi identity we obtain that the pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ has the form:

[,]	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_4$	$-2e_3$	u_3	$-u_4$	$-u_1$	u_2
e_2	0	0	0	0	u_3	u_4	$-u_1$	$-u_2$
e_3	$-2e_4$	0	0	$2e_1$	$-u_2$	u_1	$-u_4$	u_3
e_4	$2e_3$	0	$-2e_1$	0	u_4	u_3	$-u_2$	$-u_1$
u_1	$-u_3$	$-u_3$	u_2	$-u_4$	0	$-b_1e_3$	$b_1e_1 + 3b_1e_2$	b_1e_4
u_2	u_4	$-u_4$	$-u_1$	$-u_3$	b_1e_3	0	b_1e_4	$-b_1e_1 + 3b_1e_2$
u_3	u_1	u_1	u_4	u_2	$-b_1e_1 - 3b_1e_2$	$-b_1e_4$	0	$-b_1e_3$
u_4	$-u_2$	u_2	$-u_3$	u_1	$-b_1e_4$	$b_1e_1 - 3b_1e_2$	b_1e_3	0

.

Consider the following cases:

1°. $b_1 > 0$. Then the pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ is equivalent to the pair $(\bar{\mathfrak{g}}_1^\sigma, \mathfrak{g}_1^\sigma)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1^\sigma \to \bar{\mathfrak{g}}^\sigma$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(e_2) &= e_2, \\
\pi(e_3) &= e_3, \\
\pi(e_4) &= e_4, \\
\pi(u_1) &= \frac{1}{\sqrt{b_1}} u_1, \\
\pi(u_2) &= \frac{1}{\sqrt{b_1}} u_2, \\
\pi(u_3) &= \frac{1}{\sqrt{b_1}} u_3, \\
\pi(u_4) &= \frac{1}{\sqrt{b_1}} u_4.
\end{aligned}$$

2°. $b_1 < 0$. Then the pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ is equivalent to the pair $(\bar{\mathfrak{g}}_2^\sigma, \mathfrak{g}_2^\sigma)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2^\sigma \to \bar{\mathfrak{g}}^\sigma$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(e_2) &= e_2, \\
\pi(e_3) &= e_3, \\
\pi(e_4) &= e_4, \\
\pi(u_1) &= \frac{1}{\sqrt{-b_1}} u_1, \\
\pi(u_2) &= \frac{1}{\sqrt{-b_1}} u_2, \\
\pi(u_3) &= \frac{1}{\sqrt{-b_1}} u_3, \\
\pi(u_4) &= \frac{1}{\sqrt{-b_1}} u_4.
\end{aligned}$$

3°. $b_1 = 0$. Then the pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_3^\sigma, \mathfrak{g}_3^\sigma)$.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_i^\sigma \neq \dim \mathcal{D}\bar{\mathfrak{g}}_3^\sigma$, $i = 1, 2$ we see that the pairs $(\bar{\mathfrak{g}}_i^\sigma, \mathfrak{g}_i^\sigma)$ and $(\bar{\mathfrak{g}}_3^\sigma, \mathfrak{g}_3^\sigma)$ are not equivalent.

Consider the pairs $(\bar{\mathfrak{g}}_1^\sigma, \mathfrak{g}_1^\sigma)$ and $(\bar{\mathfrak{g}}_2^\sigma, \mathfrak{g}_2^\sigma)$.

Since one of them is isomorphic to $\mathfrak{su}(3)$ and the other to $\mathfrak{su}(2,1)$, we see that the pairs $(\bar{\mathfrak{g}}_1^\sigma, \mathfrak{g}_1^\sigma)$ and $(\bar{\mathfrak{g}}_2^\sigma, \mathfrak{g}_2^\sigma)$. are not equivalent.

The proof of the Proposition is complete.

The proof of the following Theorems is similar.

Theorem 3.4. *Any real form of the linear Lie algebra 3.4 is conjugate to one and only one of the following linear Lie algebras:*

$$3.4^1 \quad \begin{pmatrix} x & y & 0 & 0 \\ z & -x & 0 & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & x \end{pmatrix} \qquad 3.4^2 \quad \begin{pmatrix} 0 & y & -x & -z \\ -y & 0 & -z & x \\ x & z & 0 & y \\ z & -x & -y & 0 \end{pmatrix}$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 3.4^1 *is equivalent to one and only one pair:*

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	u_1	$-u_2$	$-u_3$	u_4
e_2	$-2e_2$	0	e_1	0	u_1	$-u_4$	0
e_3	$2e_3$	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	0	$-u_2$	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0	0
u_3	u_3	u_4	0	0	0	0	0
u_4	$-u_4$	0	u_3	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 3.4^2 *is equivalent to one and only one pair:*

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_3$	$-2e_2$	u_3	$-u_4$	$-u_1$	u_2
e_2	$-2e_3$	0	$2e_1$	$-u_2$	u_1	$-u_4$	u_3
e_3	$2e_2$	$-2e_1$	0	u_4	u_3	$-u_2$	$-u_1$
u_1	$-u_3$	u_2	$-u_4$	0	0	0	0
u_2	u_4	$-u_1$	$-u_3$	0	0	0	0
u_3	u_1	u_4	u_2	0	0	0	0
u_4	$-u_2$	$-u_3$	u_1	0	0	0	0

Theorem 3.5. *Any real form of the linear Lie algebra 3.5 is conjugate to one and only one of the following linear Lie algebras:*

$$3.5^1 \quad \begin{pmatrix} 2x & y & 0 & 0 \\ 2z & 0 & -2y & 0 \\ 0 & -z & -2x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad 3.5^2 \quad \begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & z & 0 \\ -y & -z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 3.5^1 *is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	0	0	u_1
u_2	0	$-u_1$	u_3	0	0	0	u_2
u_3	$2u_3$	$2u_2$	0	0	0	0	u_3
u_4	0	0	0	$-u_1$	$-u_2$	$-u_3$	0

2.

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	e_2	e_1	0
u_2	0	$-u_1$	u_3	$-e_2$	0	e_3	0
u_3	$2u_3$	$2u_2$	0	$-e_1$	$-e_3$	0	0
u_4	0	0	0	0	0	0	0

3.

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	$-e_2$	$-e_1$	0
u_2	0	$-u_1$	u_3	e_2	0	$-e_3$	0
u_3	$2u_3$	$2u_2$	0	e_1	e_3	0	0
u_4	0	0	0	0	0	0	0

4.

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	0	0	0
u_2	0	$-u_1$	u_3	0	0	0	0
u_3	$2u_3$	$2u_2$	0	0	0	0	0
u_4	0	0	0	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ of type 3.5^2 is equivalent to one and only one of the following pairs:

1.

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	$-u_2$	u_1	0	0
e_2	e_3	0	$-e_1$	$-u_3$	0	u_1	0
e_3	$-e_2$	e_1	0	0	$-u_3$	u_2	0
u_1	u_2	u_3	0	0	0	0	u_1
u_2	$-u_1$	0	u_3	0	0	0	u_2
u_3	0	$-u_1$	$-u_2$	0	0	0	u_3
u_4	0	0	0	$-u_1$	$-u_2$	$-u_3$	0

2.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	$-u_2$	u_1	0	0
e_2	e_3	0	$-e_1$	$-u_3$	0	u_1	0
e_3	$-e_2$	e_1	0	0	$-u_3$	u_2	0
u_1	u_2	u_3	0	0	e_1	e_2	0
u_2	$-u_1$	0	u_3	$-e_1$	0	e_3	0
u_3	0	$-u_1$	$-u_2$	$-e_2$	$-e_3$	0	0
u_4	0	0	0	0	0	0	0

3.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	$-u_2$	u_1	0	0
e_2	e_3	0	$-e_1$	$-u_3$	0	u_1	0
e_3	$-e_2$	e_1	0	0	$-u_3$	u_2	0
u_1	u_2	u_3	0	0	$-e_1$	$-e_2$	0
u_2	$-u_1$	0	u_3	e_1	0	$-e_3$	0
u_3	0	$-u_1$	$-u_2$	e_2	e_3	0	0
u_4	0	0	0	0	0	0	0

4.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	$-u_2$	u_1	0	0
e_2	e_3	0	$-e_1$	$-u_3$	0	u_1	0
e_3	$-e_2$	e_1	0	0	$-u_3$	u_2	0
u_1	u_2	u_3	0	0	0	0	0
u_2	$-u_1$	0	u_3	0	0	0	0
u_3	0	$-u_1$	$-u_2$	0	0	0	0
u_4	0	0	0	0	0	0	0

Theorem 4.3. *Any real form of the linear Lie algebra 4.3 is conjugate to one and only one linear Lie algebra:*

$$4.3^1 \quad \begin{pmatrix} x & y & 0 & t \\ z & -x & -t & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & x \end{pmatrix}$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 4.3^1 *is equivalent to one and only one of the following pairs:*

1.

$[,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	0	u_1	$-u_2$	$-u_3$	u_4
e_2	$-2e_2$	0	e_1	0	0	u_1	$-u_4$	0
e_3	$2e_3$	$-e_1$	0	0	u_2	0	0	$-u_3$
e_4	0	0	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	$-u_2$	0	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0	0	0
u_3	u_3	u_4	0	u_2	0	0	0	e_4
u_4	$-u_4$	0	u_3	$-u_1$	0	0	$-e_4$	0

2.

$[,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	0	u_1	$-u_2$	$-u_3$	u_4
e_2	$-2e_2$	0	e_1	0	0	u_1	$-u_4$	0
e_3	$2e_3$	$-e_1$	0	0	u_2	0	0	$-u_3$
e_4	0	0	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	$-u_2$	0	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0	0	0
u_3	u_3	u_4	0	u_2	0	0	0	0
u_4	$-u_4$	0	u_3	$-u_1$	0	0	0	0

Theorem 5.1. *Any real form of the linear Lie algebra* 5.1 *is conjugate to one and only one linear Lie algebra:*

$$5.1^1 \quad \begin{pmatrix} x & y & 0 & u \\ z & t & -u & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & -t \end{pmatrix}$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 5.1^1 *is equivalent to one and only one pair:*

1.

$[,]$	e_1	e_2	e_3	e_4	e_5	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	0	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	$2e_5$	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	0	u_2	0	0	$-u_3$
e_5	0	$-2e_5$	0	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	0	0	0	0
u_2	u_2	$-u_2$	$-u_1$	0	0	0	0	0	0
u_3	u_3	u_3	u_4	0	u_2	0	0	0	0
u_4	$-u_4$	u_4	0	u_3	$-u_1$	0	0	0	0

Theorem 6.1. *Any real form of the linear Lie algebra 6.1 is conjugate to one and only one of the following linear Lie algebras:*

$$6.1^1 \quad \begin{pmatrix} x & y & 0 & u \\ z & t & -u & 0 \\ 0 & v & -x & -z \\ -v & 0 & -y & -t \end{pmatrix} \qquad 6.1^2 \quad \begin{pmatrix} 0 & x & y & z \\ -x & 0 & t & u \\ -y & -t & 0 & v \\ -z & -u & -v & 0 \end{pmatrix}$$

$$6.1^3 \quad \begin{pmatrix} 0 & x & y & z \\ -x & 0 & t & u \\ -y & -t & 0 & v \\ z & u & v & 0 \end{pmatrix}$$

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 6.1^1 *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	0	0	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	$2e_5$	$-2e_6$	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	0	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	0	0	u_2	0	0	$-u_3$
e_5	0	$-2e_5$	0	0	0	$-e_2$	0	0	$-u_2$	u_1
e_6	0	$2e_6$	0	0	e_2	0	$-u_4$	u_3	0	0
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	u_4	0	$2e_5$	e_1+e_2	$2e_3$
u_2	u_2	$-u_2$	$-u_1$	0	0	$-u_3$	$-2e_5$	0	$2e_4$	$-e_1+e_2$
u_3	u_3	u_3	u_4	0	u_2	0	$-e_1-e_2$	$-2e_4$	0	$2e_6$
u_4	$-u_4$	u_4	0	u_3	$-u_1$	0	$-2e_3$	e_1-e_2	$-2e_6$	0

2.

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	0	0	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	$2e_5$	$-2e_6$	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	0	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	0	0	u_2	0	0	$-u_3$
e_5	0	$-2e_5$	0	0	0	$-e_2$	0	0	$-u_2$	u_1
e_6	0	$2e_6$	0	0	e_2	0	$-u_4$	u_3	0	0
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	u_4	0	0	0	0
u_2	u_2	$-u_2$	$-u_1$	0	0	$-u_3$	0	0	0	0
u_3	u_3	u_3	u_4	0	u_2	0	0	0	0	0
u_4	$-u_4$	u_4	0	u_3	$-u_1$	0	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 6.1^2 *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	$-e_4$	$-e_5$	e_2	e_3	0	$-u_2$	u_1	0	0
e_2	e_4	0	$-e_6$	$-e_1$	0	e_3	$-u_3$	0	u_1	0
e_3	e_5	e_6	0	0	$-e_1$	$-e_2$	$-u_4$	0	0	u_1
e_4	$-e_2$	e_1	0	0	$-e_6$	e_5	0	$-u_3$	u_2	0
e_5	$-e_3$	0	e_1	e_6	0	$-e_4$	0	$-u_4$	0	u_2
e_6	0	$-e_3$	e_2	$-e_5$	e_4	0	0	0	$-u_4$	u_3
u_1	u_2	u_3	u_4	0	0	0	0	e_1	e_2	e_3
u_2	$-u_1$	0	0	u_3	u_4	0	$-e_1$	0	e_4	e_5
u_3	0	$-u_1$	0	$-u_2$	0	u_4	$-e_2$	$-e_4$	0	e_6
u_4	0	0	$-u_1$	0	$-u_2$	$-u_3$	$-e_3$	$-e_5$	$-e_6$	0

2.

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	$-e_4$	$-e_5$	e_2	e_3	0	$-u_2$	u_1	0	0
e_2	e_4	0	$-e_6$	$-e_1$	0	e_3	$-u_3$	0	u_1	0
e_3	e_5	e_6	0	0	$-e_1$	$-e_2$	$-u_4$	0	0	u_1
e_4	$-e_2$	e_1	0	0	$-e_6$	e_5	0	$-u_3$	u_2	0
e_5	$-e_3$	0	e_1	e_6	0	$-e_4$	0	$-u_4$	0	u_2
e_6	0	$-e_3$	e_2	$-e_5$	e_4	0	0	0	$-u_4$	u_3
u_1	u_2	u_3	u_4	0	0	0	0	$-e_1$	$-e_2$	$-e_3$
u_2	$-u_1$	0	0	u_3	u_4	0	e_1	0	$-e_4$	$-e_5$
u_3	0	$-u_1$	0	$-u_2$	0	u_4	e_2	e_4	0	$-e_6$
u_4	0	0	$-u_1$	0	$-u_2$	$-u_3$	e_3	e_5	e_6	0

3.

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	$-e_4$	$-e_5$	e_2	e_3	0	$-u_2$	u_1	0	0
e_2	e_4	0	$-e_6$	$-e_1$	0	e_3	$-u_3$	0	u_1	0
e_3	e_5	e_6	0	0	$-e_1$	$-e_2$	$-u_4$	0	0	u_1
e_4	$-e_2$	e_1	0	0	$-e_6$	e_5	0	$-u_3$	u_2	0
e_5	$-e_3$	0	e_1	e_6	0	$-e_4$	0	$-u_4$	0	u_2
e_6	0	$-e_3$	e_2	$-e_5$	e_4	0	0	0	$-u_4$	u_3
u_1	u_2	u_3	u_4	0	0	0	0	0	0	0
u_2	$-u_1$	0	0	u_3	u_4	0	0	0	0	0
u_3	0	$-u_1$	0	$-u_2$	0	u_4	0	0	0	0
u_4	0	0	$-u_1$	0	$-u_2$	$-u_3$	0	0	0	0

Any isotropically faithful pair $(\bar{\mathfrak{g}}^\sigma, \mathfrak{g}^\sigma)$ *of type* 6.1^3 *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	$-e_4$	$-e_5$	e_2	e_3	0	$-u_2$	u_1	0	0
e_2	e_4	0	$-e_6$	$-e_1$	0	e_3	$-u_3$	0	u_1	0
e_3	e_5	e_6	0	0	e_1	e_2	u_4	0	0	u_1
e_4	$-e_2$	e_1	0	0	$-e_6$	e_5	0	$-u_3$	u_2	0
e_5	$-e_3$	0	$-e_1$	e_6	0	e_4	0	u_4	0	u_2
e_6	0	$-e_3$	$-e_2$	$-e_5$	$-e_4$	0	0	0	u_4	u_3
u_1	u_2	u_3	$-u_4$	0	0	0	0	e_1	e_2	$-e_3$
u_2	$-u_1$	0	0	u_3	$-u_4$	0	$-e_1$	0	e_4	$-e_5$
u_3	0	$-u_1$	0	$-u_2$	0	$-u_4$	$-e_2$	e_4	0	$-e_6$
u_4	0	0	$-u_1$	0	$-u_2$	$-u_3$	e_3	e_5	e_6	0

2.

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	$-e_4$	$-e_5$	e_2	e_3	0	$-u_2$	u_1	0	0
e_2	e_4	0	$-e_6$	$-e_1$	0	e_3	$-u_3$	0	u_1	0
e_3	e_5	e_6	0	0	e_1	e_2	u_4	0	0	u_1
e_4	$-e_2$	e_1	0	0	$-e_6$	e_5	0	$-u_3$	u_2	0
e_5	$-e_3$	0	$-e_1$	e_6	0	e_4	0	u_4	0	u_2
e_6	0	$-e_3$	$-e_2$	$-e_5$	$-e_4$	0	0	0	u_4	u_3
u_1	u_2	u_3	$-u_4$	0	0	0	0	$-e_1$	$-e_2$	e_3
u_2	$-u_1$	0	0	u_3	$-u_4$	0	e_1	0	$-e_4$	e_5
u_3	0	$-u_1$	0	$-u_2$	0	$-u_4$	e_2	$-e_4$	0	e_6
u_4	0	0	$-u_1$	0	$-u_2$	$-u_3$	$-e_3$	$-e_5$	$-e_6$	0

3.

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	$-e_4$	$-e_5$	e_2	e_3	0	$-u_2$	u_1	0	0
e_2	e_4	0	$-e_6$	$-e_1$	0	e_3	$-u_3$	0	u_1	0
e_3	e_5	e_6	0	0	e_1	e_2	u_4	0	0	u_1
e_4	$-e_2$	e_1	0	0	$-e_6$	e_5	0	$-u_3$	u_2	0
e_5	$-e_3$	0	$-e_1$	e_6	0	e_4	0	u_4	0	u_2
e_6	0	$-e_3$	$-e_2$	$-e_5$	$-e_4$	0	0	0	u_4	u_3
u_1	u_2	u_3	$-u_4$	0	0	0	0	0	0	0
u_2	$-u_1$	0	0	u_3	$-u_4$	0	0	0	0	0
u_3	0	$-u_1$	0	$-u_2$	0	$-u_4$	0	0	0	0
u_4	0	0	$-u_1$	0	$-u_2$	$-u_3$	0	0	0	0

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